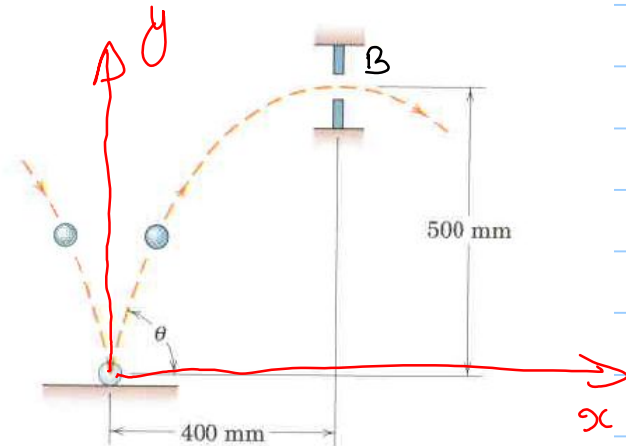


2/89 To meet design criteria, small ball bearings must bounce through an opening of limited size at the top of their trajectory when rebounding from a heavy plate as shown. Calculate the angle θ made by the rebound velocity with the horizontal and the velocity v of the balls as they pass through the opening.

Ans. $\theta = 68.2^\circ$, $v = 1.253$ m/s



$$B \begin{cases} 400 \\ 500 \end{cases}$$

$$y = \frac{-g}{2v_0^2 \cos^2 \theta} x^2 + \tan \theta x$$

$$0,5 = \frac{-9,81}{2v_0^2 \cos^2 \theta} x^2 + \tan \theta \times 0,4$$

$$\begin{cases} x = v_0 \cos \theta t \\ y = -\frac{1}{2} g t^2 + v_0 \sin \theta t \end{cases}$$

$$\rightarrow \begin{cases} v_x = v_0 \cos \theta \\ v_y = -gt + v_0 \sin \theta \end{cases}$$

$$\text{at } B \rightarrow v_y = 0 \rightarrow 0 = -gt + v_0 \sin \theta \rightarrow t = \frac{v_0 \sin \theta}{g} \quad \text{Eq. 1}$$

$$\text{at } C, \quad 0,4 = v_0 \cos \theta \times \frac{v_0 \sin \theta}{g} \Rightarrow 0,4 \times 9,81 = v_0^2 \sin \theta \cos \theta$$

$$v_0^2 = \frac{0,4 \times 9,81}{\sin \theta \cos \theta} \quad 0,5 = \frac{-9,81 \times 0,4^2}{2 \times \frac{0,4 \times 9,81}{\sin \theta \cos \theta} \times \cos^2 \theta} + \tan \theta \times 0,4$$

$$0,5 = -0,2 \tan \theta + 0,4 \tan \theta \rightarrow \tan \theta = 2,5 \rightarrow \theta = 68,19$$

$$v_0^2 = \frac{0,4 \times 9,81}{2 \cdot 68,2 \cos 68,2} \rightarrow v_0^2 = \frac{3,924}{0,345} \rightarrow v_0 = 3,372 \text{ m/sec}$$

$$\text{Eq. 2, } v = v_0 \cos \theta = 3,372 \cos 68,2 = 1,252 \text{ m/sec}$$

2/5 NORMAL AND TANGENTIAL COORDINATES (n - t)

As we mentioned in Art. 2/1, one of the common descriptions of curvilinear motion uses path variables, which are measurements made along the tangent t and normal n to the path of the particle. These coordinates provide a very natural description for curvilinear motion and are frequently the most direct and convenient coordinates to use. The n - and t -coordinates are considered to move along the path with the particle, as seen in Fig. 2/9 where the particle advances from A to B to C . The positive direction for n at any position is always taken toward the center of curvature of the path. As seen from Fig. 2/9, the positive n -direction will shift from one side of the curve to the other side if the curvature changes direction.

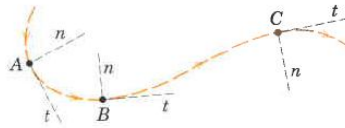
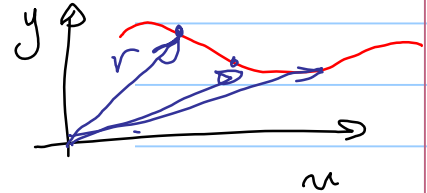


Figure 2/9



Velocity and Acceleration

We now use the coordinates n and t to describe the velocity \mathbf{v} and acceleration \mathbf{a} which were introduced in Art. 2/3 for the curvilinear motion of a particle. For this purpose, we introduce unit vectors \mathbf{e}_n in the n -direction and \mathbf{e}_t in the t -direction, as shown in Fig. 2/10a for the position of the particle at point A on its path. During a differential increment of time dt , the particle moves a differential distance ds along the curve from A to A' . With the radius of curvature of the path at this position designated by ρ , we see that $ds = \rho d\beta$, where β is in radians. It is unnecessary to consider the differential change in ρ between A and A' because a higher-order term would be introduced which disappears in the limit. Thus, the magnitude of the velocity can be written $v = ds/dt = \rho d\beta/dt$, and we can write the velocity as the vector

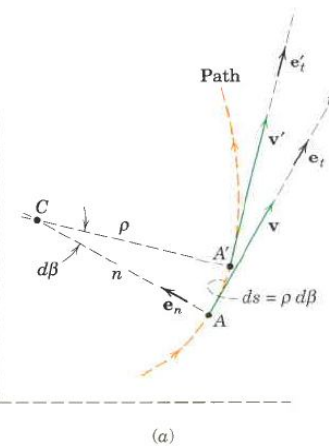
$$\mathbf{v} = v\mathbf{e}_t = \rho\dot{\beta}\mathbf{e}_t \quad (2/7)$$

The acceleration \mathbf{a} of the particle was defined in Art. 2/3 as $\mathbf{a} = d\mathbf{v}/dt$, and we observed from Fig. 2/5 that the acceleration is a vector which reflects both the change in magnitude and the change in direction of \mathbf{v} . We now differentiate \mathbf{v} in Eq. 2/7 by applying the ordinary rule for the differentiation of the product of a scalar and a vector* and get

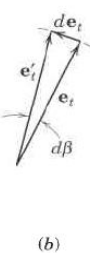
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\mathbf{e}_t)}{dt} = v\dot{\mathbf{e}}_t + \dot{v}\mathbf{e}_t \quad (2/8)$$

where the unit vector \mathbf{e}_t now has a nonzero derivative because its direction changes.

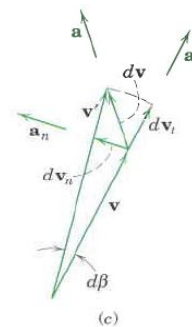
To find $\dot{\mathbf{e}}_t$ we analyze the change in \mathbf{e}_t during a differential increment of motion as the particle moves from A to A' in Fig. 2/10a. The unit vector \mathbf{e}_t correspondingly changes to \mathbf{e}_t' , and the vector difference $d\mathbf{e}_t$ is shown in part b of the figure. The vector $d\mathbf{e}_t$ in the limit has a magnitude equal to the length of the arc $|\mathbf{e}_t| d\beta = d\beta$ obtained by swinging the unit vector \mathbf{e}_t through the angle $d\beta$ expressed in radians.



(a)



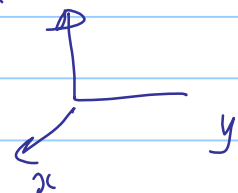
(b)



(c)

Figure 2/10

$$\begin{aligned} \mathbf{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \dot{\mathbf{r}} &= \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \end{aligned}$$



*See Art. C/7 of Appendix C.

The direction of $d\mathbf{e}_t$ is given by \mathbf{e}_n . Thus, we can write $d\mathbf{e}_t = \mathbf{e}_n d\beta$. Dividing by $d\beta$ gives

$$\frac{d\mathbf{e}_t}{d\beta} = \mathbf{e}_n$$

Dividing by dt gives $d\mathbf{e}_t/dt = (d\beta/dt)\mathbf{e}_n$, which can be written

$$\dot{\mathbf{e}}_t = \dot{\beta}\mathbf{e}_n \quad (2/9)$$

With the substitution of Eq. 2/9 and $\dot{\beta}$ from the relation $v = \rho\dot{\beta}$, Eq. 2/8 for the acceleration becomes

$$\mathbf{a} = \frac{v^2}{\rho}\mathbf{e}_n + \dot{v}\mathbf{e}_t \quad (2/10)$$

where

$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$

$$a_t = \dot{v} = \ddot{s}$$

$$a = \sqrt{a_n^2 + a_t^2}$$

We may also note that $a_t = \dot{v} = d(\rho\dot{\beta})/dt = \rho\ddot{\beta} + \dot{\rho}\dot{\beta}$. This relation, however, finds little use because we seldom have reason to compute $\dot{\rho}$.

Geometric Interpretation

Full understanding of Eq. 2/10 comes only when we clearly see the geometry of the physical changes it describes. Figure 2/10c shows the velocity vector \mathbf{v} when the particle is at A and \mathbf{v}' when it is at A' . The vector change in the velocity is $d\mathbf{v}$, which establishes the direction of the acceleration \mathbf{a} . The n -component of $d\mathbf{v}$ is labeled $d\mathbf{v}_n$, and in the limit its magnitude equals the length of the arc generated by swinging the vector \mathbf{v} as a radius through the angle $d\beta$. Thus, $|d\mathbf{v}_n| = v d\beta$ and the n -component of acceleration is $a_n = |d\mathbf{v}_n|/dt = v(d\beta/dt) = v\dot{\beta}$ as before. The t -component of $d\mathbf{v}$ is labeled $d\mathbf{v}_t$, and its magnitude is simply the change dv in the magnitude or length of the velocity vector. Therefore, the t -component of acceleration is $a_t = dv/dt = \dot{v} = \ddot{s}$ as before. The acceleration vectors resulting from the corresponding vector changes in velocity are shown in Fig. 2/10c.

It is especially important to observe that the normal component of acceleration a_n is *always directed toward the center of curvature C* . The tangential component of acceleration, on the other hand, will be in the positive t -direction of motion if the speed v is increasing and in the negative t -direction if the speed is decreasing. In Fig. 2/11 are shown schematic representations of the variation in the acceleration vector for a particle moving from A to B with (a) increasing speed and (b) decreasing speed. At an inflection point on the curve, the normal acceleration v^2/ρ goes to zero because ρ becomes infinite.

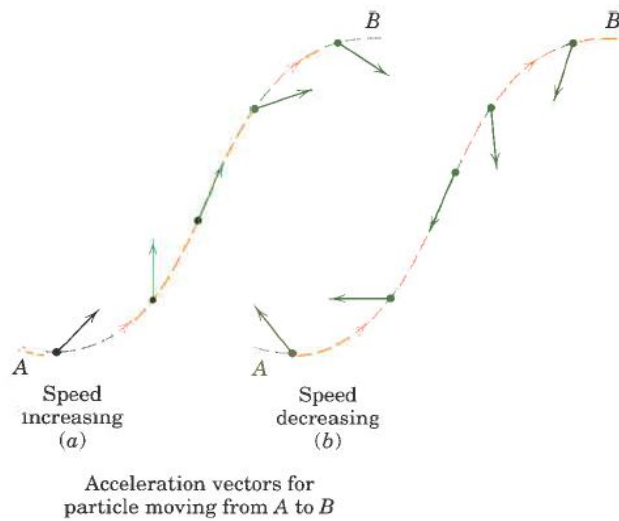


Figure 2/11

Circular Motion

Circular motion is an important special case of plane curvilinear motion where the radius of curvature ρ becomes the constant radius r of the circle and the angle β is replaced by the angle θ measured from any convenient radial reference to OP , Fig. 2/12. The velocity and the acceleration components for the circular motion of the particle P become

$$\begin{aligned} v &= r\dot{\theta} \\ a_n &= v^2/r = r\dot{\theta}^2 = v\dot{\theta} \\ a_t &= \dot{v} = r\ddot{\theta} \end{aligned} \quad (2/11)$$

We find repeated use for Eqs. 2/10 and 2/11 in dynamics, so these relations and the principles behind them should be mastered.

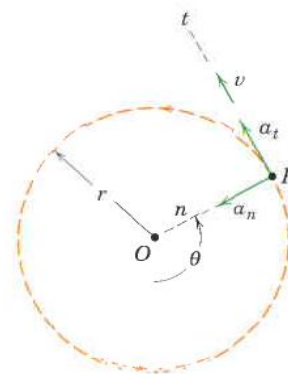


Figure 2/12



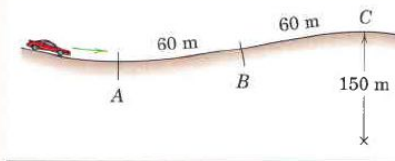
Courtesy of Ken Hester

An example of uniform circular motion is this car moving with constant speed around a skidpad, which is a circular roadway with a diameter of about 200 feet.



Sample Problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom *A* of the dip and 50 km/h at the top *C* of the hump, which is 120 m along the road from *A*. If the passengers experience a total acceleration of 3 m/s^2 at *A* and if the radius of curvature of the hump at *C* is 150 m, calculate (a) the radius of curvature ρ at *A*, (b) the acceleration at the inflection point *B*, and (c) the total acceleration at *C*.



Solution. The dimensions of the car are small compared with those of the path, so we will treat the car as a particle. The velocities are

$$v_A = \left(100 \frac{\text{km}}{\text{h}}\right) \left(\frac{1 \text{ h}}{3600 \text{ s}}\right) \left(1000 \frac{\text{m}}{\text{km}}\right) = 27.8 \text{ m/s}$$

$$v_C = 50 \frac{1000}{3600} = 13.89 \text{ m/s}$$

We find the constant deceleration along the path from

$$\left[\int v \, dv = \int a_t \, ds \right] \quad \int_{v_A}^{v_C} v \, dv = a_t \int_0^s ds$$

$$a_t = \frac{1}{2s} (v_C^2 - v_A^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41 \text{ m/s}^2$$

(a) Condition at A. With the total acceleration given and a_t determined, we can easily compute a_n and hence ρ from

$$[a^2 = a_n^2 + a_t^2] \quad a_n^2 = 3^2 - (2.41)^2 = 3.19 \quad a_n = 1.785 \text{ m/s}^2$$

$$[a_n = v^2/\rho] \quad \rho = v^2/a_n = (27.8)^2/1.785 = 432 \text{ m} \quad \text{Ans.}$$

(b) Condition at B. Since the radius of curvature is infinite at the inflection point, $a_n = 0$ and

$$a = a_t = -2.41 \text{ m/s}^2 \quad \text{Ans.}$$

(c) Condition at C. The normal acceleration becomes

$$[a_n = v^2/\rho] \quad a_n = (13.89)^2/150 = 1.286 \text{ m/s}^2$$

With unit vectors \mathbf{e}_n and \mathbf{e}_t in the n - and t -directions, the acceleration may be written

$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t \text{ m/s}^2$$

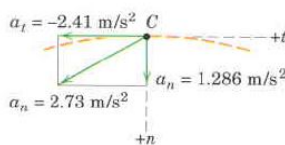
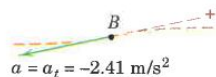
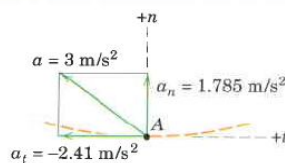
where the magnitude of \mathbf{a} is

$$[a = \sqrt{a_n^2 + a_t^2}] \quad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73 \text{ m/s}^2 \quad \text{Ans.}$$

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

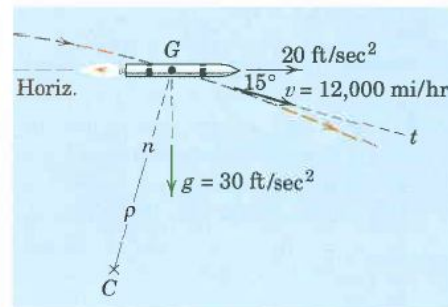
Helpful Hint

① Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.



Sample Problem 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of 20 ft/sec^2 , and the downward acceleration component is the acceleration due to gravity at that altitude, which is $g = 30 \text{ ft/sec}^2$. At the instant represented, the velocity of the mass center G of the rocket along the 15° direction of its trajectory is $12,000 \text{ mi/hr}$. For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed v is increasing, (c) the angular rate $\dot{\beta}$ of the radial line from G to the center of curvature C , and (d) the vector expression for the total acceleration \mathbf{a} of the rocket.



Solution. We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use n - and t -coordinates to describe the motion of G . The n - and t -components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their n - and t -components and then combining. From the figure we get

$$a_n = 30 \cos 15^\circ - 20 \sin 15^\circ = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15^\circ + 20 \cos 15^\circ = 27.1 \text{ ft/sec}^2$$

(a) We may now compute the radius of curvature from

$$[a_n = v^2/\rho] \quad \rho = \frac{v^2}{a_n} = \frac{[(12,000)(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft} \quad \checkmark$$

Ans.

(b) The rate at which v is increasing is simply the t -component of acceleration.

$$[\dot{v} = a_t] \quad \dot{v} = 27.1 \text{ ft/sec}^2 \quad \text{Ans.}$$

(c) The angular rate $\dot{\beta}$ of line GC depends on v and ρ and is given by

$$[v = \rho\dot{\beta}] \quad \dot{\beta} = v/\rho = \frac{12,000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec} \quad \text{Ans.}$$

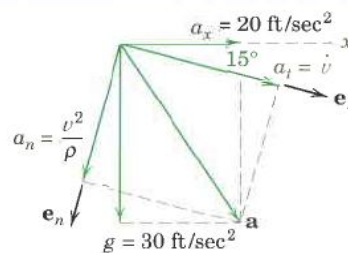
(d) With unit vectors \mathbf{e}_n and \mathbf{e}_t for the n - and t -directions, respectively, the total acceleration becomes

$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2 \quad \text{Ans.}$$

Helpful Hints

① Alternatively, we could find the resultant acceleration and then resolve it into n - and t -components,

② To convert from mi/hr to ft/sec, multiply by $\frac{5280 \text{ ft/mi}}{3600 \text{ sec/hr}} = \frac{44 \text{ ft/sec}}{30 \text{ mi/hr}}$ which is easily remembered, as 30 mi/hr is the same as 44 ft/sec .



2/105 The car travels at a constant speed from the bottom A of the dip to the top B of the hump. If the radius of curvature of the road at A is $\rho_A = 120$ m and the car acceleration at A is $0.4g$, determine the car speed v . If the acceleration at B must be limited to $0.25g$, determine the minimum radius of curvature ρ_B of the road at B .

Ans. $v = 21.6$ m/s, $\rho_B = 190.4$ m



$$\rho_A = 120 \text{ m}$$

$$a_A = 0.4g$$

$$a_B = 0.25g$$

$$v_B, \rho_B$$

$$a_t = 0$$

$$a_{nA} = 0.4g = \frac{v^2}{\rho_A} \rightarrow 0.4 \times 9.81 = \frac{v^2}{(120 - 0.6)}$$

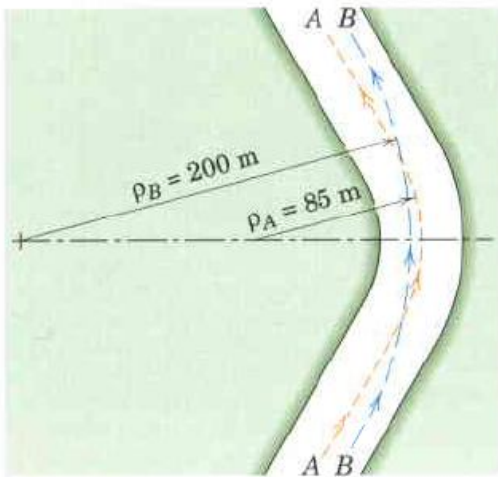
$$v = \sqrt{0.4 \times 9.81 \times (120 - 0.6)} = 21.64 \text{ m/s}$$

$$a_B \rightarrow a_B = \frac{v_B^2}{\rho_B + 0.6} \rightarrow 0.25 \times 9.81 = \frac{21.64^2}{\rho_B + 0.6}$$

$$\rho_B = \frac{21.64^2}{0.25 \times 9.81} - 0.6 = 190.3 \text{ m}$$

2/109 The figure shows two possible paths for negotiating an unbanked turn on a horizontal portion of a race course. Path A-A follows the centerline of the road and has a radius of curvature $\rho_A = 85$ m, while path B-B uses the width of the road to good advantage in increasing the radius of curvature to $\rho_B = 200$ m. If the drivers limit their speeds in their curves so that the lateral acceleration does not exceed $0.8g$, determine the maximum speed for each path.

Ans. $v_A = 25.8$ m/s, $v_B = 39.6$ m/s



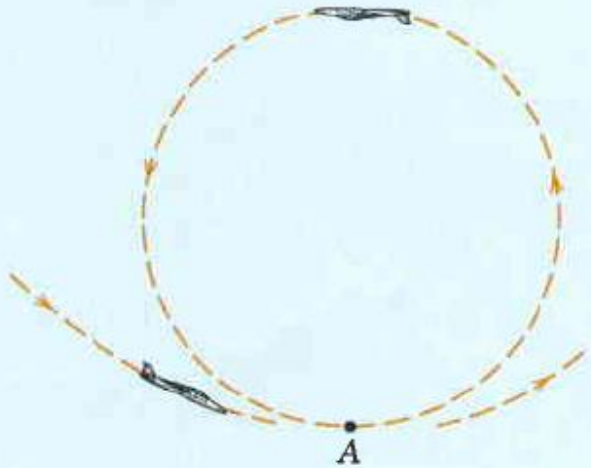
$$a_n \leq 0.8 \times 9.81 = \frac{v^2}{\rho}$$

$$v_A \leq \sqrt{0.8 \times 9.81 \times 85}$$

$$v_B \leq \sqrt{0.8 \times 9.81 \times 200}$$

2/115 At the bottom A of the vertical inside loop, the magnitude of the total acceleration of the airplane is $3g$. If the airspeed is 800 km/h and is increasing at the rate of $20 \text{ km/h per second}$, calculate the radius of curvature ρ of the path at A.

Ans. $\rho = 1709 \text{ m}$



Problem 2/115

$$a_{tot} = 3 \times 9,81$$

$$v = 800 \text{ km/h}$$

$$a_t = 20 \text{ km/h/sec}$$

$$\rho$$

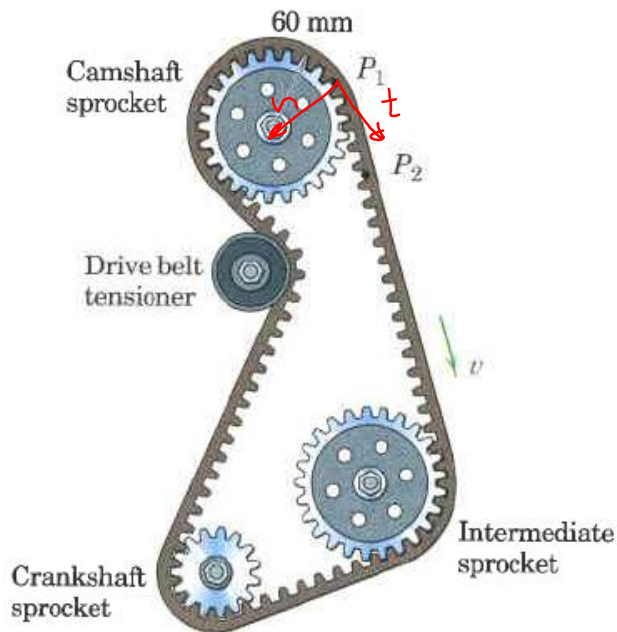
$$a_t = 20 / 3,6 \text{ m/sec}^2$$

$$a_t = 5,56 \text{ m/sec}^2$$

$$a_n^2 = a_{tot}^2 - a_t^2 \rightarrow a_n = \sqrt{(3 \times 9,81)^2 - 5,56^2}$$

$$a_n = \frac{v^2}{\rho} \rightarrow \rho = \frac{(800 / 3,6)^2}{a_n} \quad \text{m}$$

2/122 The design of a camshaft-drive system of a four-cylinder automobile engine is shown. As the engine is revved up, the belt speed v changes uniformly from 3 m/s to 6 m/s over a two-second interval. Calculate the magnitudes of the accelerations of points P_1 and P_2 halfway through this time interval.



$$3 \text{ m/sec} \rightarrow 6 \text{ m/sec} \\ \text{at } 2 \text{ sec}$$

$$a_t = \checkmark \quad a_t = 1,5 \text{ m/sec}^2$$

$$a_{P1} = ? \quad a_n = \frac{v^2}{r}$$

$$a_{P2} = ? \quad a_{P1} = \sqrt{a_n^2 + a_t^2}$$

$$a_{P2} = a_t = 1,5 \text{ m/sec}^2$$