دینامیک - جلسه اول: - بهار ۹۸

1 Introduction to Dynamics

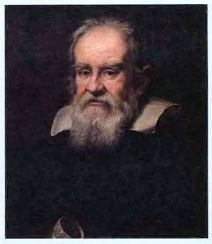
Chapter Outline

1/1 History and Modern Applications

Dynamics is that branch of mechanics which deals with the motion of bodies under the action of forces. The study of dynamics in engineering usually follows the study of statics, which deals with the effects of forces on bodies at rest. Dynamics has two distinct parts: *kinematics*, which is the study of motion without reference to the forces which cause motion, and *kinetics*, which relates the action of forces on bodies to their resulting motions. A thorough comprehension of dynamics will provide one of the most useful and powerful tools for analysis in engineering.

History of Dynamics

Dynamics is a relatively recent subject compared with statics. The beginning of a rational understanding of dynamics is credited to Galileo (1564–1642), who made careful observations concerning bodies in free fall, motion on an inclined plane, and motion of the pendulum. He was largely responsible for bringing a scientific approach to the investigation of physical problems. Galileo was continually under severe criticism for refusing to accept the established beliefs of his day, such as the philosophies of Aristotle which held, for example, that heavy bodies fall more rapidly than light bodies. The lack of accurate means for the measurement of time was a severe handicap to Galileo, and further significant development in dynamics awaited the invention of the pendulum clock by Huygens in 1657.

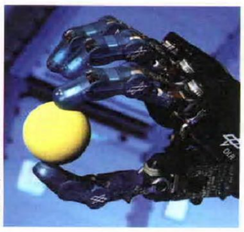


Galileo Galilei

Portrait of Galileo Galilei (1564–1642) (oil on canvas), Sustermans, Justus (1597–1681) (school of)/Galleria Palatina, Florence, Italy/Bridgeman Art Library

Newton (1642–1727), guided by Galileo's work, was able to make an accurate formulation of the laws of motion and, thus, to place dynamics

on a sound basis. Newton's famous work was published in the first edition of his *Principia*,* which is generally recognized as one of the greatest of all recorded contributions to knowledge. In addition to stating the laws governing the motion of a particle, Newton was the first to correctly formulate the law of universal gravitation. Although his mathematical description was accurate, he felt that the concept of remote transmission of gravitational force without a supporting medium was an absurd notion. Following Newton's time, important contributions to mechanics were made by Euler, D'Alembert, Lagrange, Laplace, Poinsot, Coriolis, Einstein, and others.



Dieter Klein/laif/Redux

Robot hand

Applications of Dynamics

Only since machines and structures have operated with high speeds and appreciable accelerations has it been necessary to make calculations based on the principles of dynamics rather than on the principles of statics. The rapid technological developments of the present day require increasing application of the principles of mechanics, particularly dynamics. These principles are basic to the analysis and design of moving structures, to fixed structures subject to shock loads, to robotic devices, to automatic control systems, to rockets, missiles, and spacecraft, to ground and air transportation vehicles, to electron ballistics of electrical devices, and to machinery of all types such as turbines, pumps, reciprocating engines, hoists, machine tools, etc.

Students with interests in one or more of these and many other activities will constantly need to apply the fundamental principles of dynamics.

1/2 Basic Concepts

The concepts basic to mechanics were set forth in Art. 1/2 of *Vol. 1 Statics*. They are summarized here along with additional comments of special relevance to the study of dynamics.

Space is the geometric region occupied by bodies. Position in space is determined relative to some geometric reference system by means of linear and angular measurements. The basic frame of reference for the laws of Newtonian mechanics is the *primary inertial system* or *astronomical frame of reference*, which is an imaginary set of rectangular axes assumed to have no translation or rotation in space. Measurements show that the laws of Newtonian mechanics are valid for this reference system as long as any velocities involved are negligible compared with the speed of light, which is 300 000 km/s or 186,000 mi/sec. Measurements made with respect to this reference are said to be *absolute*, and this reference system may be considered "fixed" in space.

A reference frame attached to the surface of the earth has a somewhat complicated motion in the primary system, and a correction to the basic equations of mechanics must be applied for measurements made

*The original formulations of Sir Isaac Newton may be found in the translation of his *Principia* (1687), revised by F. Cajori, University of California Press, 1934.

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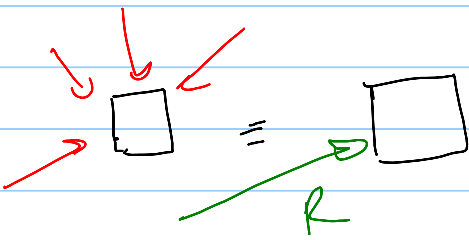

$$\sum \vec{F} = \frac{dmv}{dt} = m\frac{dv}{dt} = m\vec{a}$$

بررسی علم حرکت اجسام با عامل به وجود آورنده آن سینتیک

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relative to the reference frame of the earth. In the calculation of rocket and space-flight trajectories, for example, the absolute motion of the earth becomes an important parameter. For most engineering problems involving machines and structures which remain on the surface of the earth, the corrections are extremely small and may be neglected. For these problems the laws of mechanics may be applied directly with measurements made relative to the earth, and in a practical sense such measurements will be considered *absolute*.

Time is a measure of the succession of events and is considered an absolute quantity in Newtonian mechanics.

Mass is the quantitative measure of the inertia or resistance to change in motion of a body. Mass may also be considered as the quantity of matter in a body as well as the property which gives rise to gravitational attraction.

Force is the vector action of one body on another. The properties of forces have been thoroughly treated in *Vol. 1 Statics*.

A ***particle*** is a body of negligible dimensions. When the dimensions of a body are irrelevant to the description of its motion or the action of forces on it, the body may be treated as a particle. An airplane, for example, may be treated as a particle for the description of its flight path.

A ***rigid body*** is a body whose changes in shape are negligible compared with the overall dimensions of the body or with the changes in position of the body as a whole. As an example of the assumption of rigidity, the small flexural movement of the wing tip of an airplane flying through turbulent air is clearly of no consequence to the description of the motion of the airplane as a whole along its flight path. For this purpose, then, the treatment of the airplane as a rigid body is an acceptable approximation. On the other hand, if we need to examine the internal stresses in the wing structure due to changing dynamic loads, then the deformation characteristics of the structure would have to be examined, and for this purpose the airplane could no longer be considered a rigid body.

Vector and ***scalar*** quantities have been treated extensively in *Vol. 1 Statics*, and their distinction should be perfectly clear by now. Scalar quantities are printed in lightface italic type, and vectors are shown in boldface type. Thus, V denotes the scalar magnitude of the vector $\mathbf{V}$. It is important that we use an identifying mark, such as an underline $\underline{V}$, for all handwritten vectors to take the place of the boldface designation in print. For two nonparallel vectors recall, for example, that $\mathbf{V}_1 + \mathbf{V}_2$ and $V_1 + V_2$ have two entirely different meanings.

We assume that you are familiar with the geometry and algebra of vectors through previous study of statics and mathematics. Students who need to review these topics will find a brief summary of them in Appendix C along with other mathematical relations which find frequent use in mechanics. Experience has shown that the geometry of mechanics is often a source of difficulty for students. Mechanics by its very nature is geometrical, and students should bear this in mind as they review their mathematics. In addition to vector algebra, dynamics requires the use of vector calculus, and the essentials of this topic will be developed in the text as they are needed.

Dynamics involves the frequent use of time derivatives of both vectors and scalars. As a notational shorthand, a dot over a symbol will frequently be used to indicate a derivative with respect to time. Thus, $\dot{x}$ means dx/dt and $\ddot{x}$ stands for d^2x/dt^2.

1/3 Newton's Laws

Newton's three laws of motion, stated in Art. 1/4 of *Vol. 1 Statics*, are restated here because of their special significance to dynamics. In modern terminology they are:

Law I. A particle remains at rest or continues to move with uniform velocity (in a straight line with a constant speed) if there is no unbalanced force acting on it.

Law II. The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.*

Law III. The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear.

These laws have been verified by countless physical measurements. The first two laws hold for measurements made in an absolute frame of reference, but are subject to some correction when the motion is measured relative to a reference system having acceleration, such as one attached to the surface of the earth.

Newton's second law forms the basis for most of the analysis in dynamics. For a particle of mass m subjected to a resultant force $\mathbf{F}$, the law may be stated as

$$\boxed{\mathbf{F} = m\mathbf{a}} \tag{1/1}$$

where $\mathbf{a}$ is the resulting acceleration measured in a nonaccelerating frame of reference. Newton's first law is a consequence of the second law since there is no acceleration when the force is zero, and so the particle is either at rest or is moving with constant velocity. The third law constitutes the principle of action and reaction with which you should be thoroughly familiar from your work in statics.

1/4 Units

Both the International System of metric units (SI) and the U.S. customary system of units are defined and used in *Vol. 2 Dynamics*, although a stronger emphasis is placed on the metric system because it is replacing the U.S. customary system. However, numerical conversion from one system to the other will often be needed in U.S. engineering

*To some it is preferable to interpret Newton's second law as meaning that the resultant force acting on a particle is proportional to the time rate of change of momentum of the particle and that this change is in the direction of the force. Both formulations are equally correct when applied to a particle of constant mass.

practice for some years to come. To become familiar with each system, it is necessary to think directly in that system. Familiarity with the new system cannot be achieved simply by the conversion of numerical results from the old system.

Tables defining the SI units and giving numerical conversions between U.S. customary and SI units are included inside the front cover of the book. Charts comparing selected quantities in SI and U.S. customary units are included inside the back cover of the book to facilitate conversion and to help establish a feel for the relative size of units in both systems.

The four fundamental quantities of mechanics, and their units and symbols for the two systems, are summarized in the following table:

QUANTITY	DIMENSIONAL SYMBOL	SI UNITS			U.S. CUSTOMARY UNITS		
			UNIT	SYMBOL		UNIT	SYMBOL
Mass	M	Base units	kilogram	kg		slug	—
Length	L	Base units	meter*	m	Base units	foot	ft
Time	T	Base units	second	s	Base units	second	sec
Force	F		newton	N	Base units	pound	lb

*Also spelled *metre*.

As shown in the table, in SI the units for mass, length, and time are taken as base units, and the units for force are derived from Newton's second law of motion, Eq. 1/1. In the U.S. customary system the units for force, length, and time are base units and the units for mass are derived from the second law.

The SI system is termed an *absolute* system because the standard for the base unit kilogram (a platinum-iridium cylinder kept at the International Bureau of Standards near Paris, France) is independent of the gravitational attraction of the earth. On the other hand, the U.S. customary system is termed a *gravitational* system because the standard for the base unit pound (the weight of a standard mass located at sea level and at a latitude of 45°) requires the presence of the gravitational field of the earth. This distinction is a fundamental difference between the two systems of units.

In SI units, by definition, one newton is that force which will give a one-kilogram mass an acceleration of one meter per second squared. In the U.S. customary system a 32.1740-pound mass (1 slug) will have an acceleration of one foot per second squared when acted on by a force of one pound. Thus, for each system we have from Eq. 1/1

Courtesy Bureau International des Poids et Mesures, France

The standard kilogram

SI UNITS	U.S. CUSTOMARY UNITS
$(1\ \text{N}) = (1\ \text{kg})(1\ \text{m/s}^2)$	$(1\ \text{lb}) = (1\ \text{slug})(1\ \text{ft/sec}^2)$
$\text{N} = \text{kg}\cdot\text{m/s}^2$	$\text{slug} = \text{lb}\cdot\text{sec}^2/\text{ft}$

In SI units, the kilogram should be used *exclusively* as a unit of mass and *never* force. Unfortunately, in the MKS (meter, kilogram, second) gravitational system, which has been used in some countries for many years, the kilogram has been commonly used both as a unit of force and as a unit of mass.

In U.S. customary units, the pound is unfortunately used both as a unit of force (lbf) and as a unit of mass (lbm). The use of the unit lbm is especially prevalent in the specification of the thermal properties of liquids and gases. The lbm is the amount of mass which weighs 1 lbf under standard conditions (at a latitude of 45° and at sea level). In order to avoid the confusion which would be caused by the use of two units for mass (slug and lbm), in this textbook we use almost exclusively the unit slug for mass. This practice makes dynamics much simpler than if the lbm were used. In addition, this approach allows us to use the symbol lb to always mean pound force.

Additional quantities used in mechanics and their equivalent base units will be defined as they are introduced in the chapters which follow. However, for convenient reference these quantities are listed in one place in the first table inside the front cover of the book.

Professional organizations have established detailed guidelines for the consistent use of SI units, and these guidelines have been followed throughout this book. The most essential ones are summarized inside the front cover, and you should observe these rules carefully.

1/5 GRAVITATION

Newton's law of gravitation, which governs the mutual attraction between bodies, is

$$\boxed{F = G\frac{m_1 m_2}{r^2}} \qquad \textbf{(1/2)}$$

where F = the mutual force of attraction between two particles

G = a universal constant called the *constant of gravitation*

m_1, m_2 = the masses of the two particles

r = the distance between the centers of the particles

The value of the gravitational constant obtained from experimental data is $G = 6.673(10^{-11})\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$. Except for some spacecraft applications, the only gravitational force of appreciable magnitude in engineering is the force due to the attraction of the earth. It was shown in *Vol. 1 Statics*, for example, that each of two iron spheres 100 mm in diameter is attracted to the earth with a gravitational force of 37.1 N, which is called its *weight*, but the force of mutual attraction between them if they are just touching is only 0.000 000 095 1 N.

Because the gravitational attraction or weight of a body is a force, it should always be expressed in force units, newtons (N) in SI units and pounds force (lb) in U.S. customary units. To avoid confusion, the word "weight" in this book will be restricted to mean the force of gravitational attraction.

Effect of Altitude

The force of gravitational attraction of the earth on a body depends on the position of the body relative to the earth. If the earth were a perfect homogeneous sphere, a body with a mass of exactly 1 kg would be attracted to the earth by a force of 9.825 N on the surface of the earth, 9.822 N at an altitude of 1 km, 9.523 N at an altitude of 100 km, 7.340 N at an altitude of 1000 km, and 2.456 N at an altitude equal to the mean radius of the earth, 6371 km. Thus the variation in gravitational attraction of high-altitude rockets and spacecraft becomes a major consideration.

Every object which falls in a vacuum at a given height near the surface of the earth will have the same acceleration g, regardless of its mass. This result can be obtained by combining Eqs. 1/1 and 1/2 and canceling the term representing the mass of the falling object. This combination gives

$$g = \frac{Gm_e}{R^2}$$

$F = G \cdot \frac{\cancel{m} M_e}{r^2}$

$w = \cancel{m} g$

$\boxed{g = \frac{G M_e}{R^2}}$

where m_e is the mass of the earth and R is the radius of the earth.* The mass m_e and the mean radius R of the earth have been found through experimental measurements to be $5.976(10^{24})$ kg and $6.371(10^6)$ m, respectively. These values, together with the value of G already cited, when substituted into the expression for g, give a mean value of $g = 9.825 \text{ m/s}^2$.

The variation of g with altitude is easily determined from the gravitational law. If g_0 represents the absolute acceleration due to gravity at sea level, the absolute value at an altitude h is

$$g = g_0 \frac{R^2}{(R + h)^2}$$

where R is the radius of the earth.

Effect of a Rotating Earth

The acceleration due to gravity as determined from the gravitational law is the acceleration which would be measured from a set of axes whose origin is at the center of the earth but which does not rotate with the earth. With respect to these "fixed" axes, then, this value may be termed the *absolute* value of g. Because the earth rotates, the acceleration of a freely falling body as measured from a position attached to the surface of the earth is slightly less than the absolute value.

Accurate values of the gravitational acceleration as measured relative to the surface of the earth account for the fact that the earth is a rotating oblate spheroid with flattening at the poles. These values may

*It can be proved that the earth, when taken as a sphere with a symmetrical distribution of mass about its center, may be considered a particle with its entire mass concentrated at its center.

be calculated to a high degree of accuracy from the 1980 International Gravity Formula, which is

$$g = 9.780\ 327(1 + 0.005\ 279 \sin^2 \gamma + 0.000\ 023 \sin^4 \gamma + \cdots)$$

where γ is the latitude and g is expressed in meters per second squared. The formula is based on an ellipsoidal model of the earth and also accounts for the effect of the rotation of the earth.

The absolute acceleration due to gravity as determined for a nonrotating earth may be computed from the relative values to a close approximation by adding $3.382(10^{-2}) \cos^2\gamma$ m/s^2, which removes the effect of the rotation of the earth. The variation of both the absolute and the relative values of g with latitude is shown in Fig. 1/1 for sea-level conditions.*

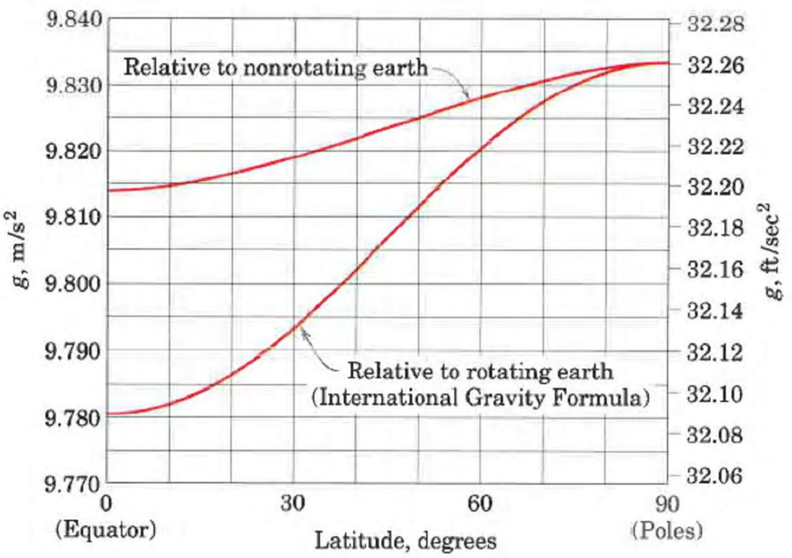


Figure 1/1

Standard Value of *g*

The standard value which has been adopted internationally for the gravitational acceleration relative to the rotating earth at sea level and at a latitude of 45° is 9.806 65 m/s^2 or 32.1740 ft/sec^2. This value differs very slightly from that obtained by evaluating the International Gravity Formula for $\gamma = 45°$. The reason for the small difference is that the earth is not exactly ellipsoidal, as assumed in the formulation of the International Gravity Formula.

The proximity of large land masses and the variations in the density of the crust of the earth also influence the local value of g by a small but detectable amount. In almost all engineering applications near the surface of the earth, we can neglect the difference between the absolute and relative values of the gravitational acceleration, and the effect of local

*You will be able to derive these relations for a spherical earth after studying relative motion in Chapter 3.

variations. The values of 9.81 m/s^2 in SI units and 32.2 ft/sec^2 in U.S. customary units are used for the sea-level value of g.

Apparent Weight

The gravitational attraction of the earth on a body of mass m may be calculated from the results of a simple gravitational experiment. The body is allowed to fall freely in a vacuum, and its absolute acceleration is measured. If the gravitational force of attraction or true weight of the body is W, then, because the body falls with an absolute acceleration g, Eq. 1/1 gives

$$\boxed{\mathbf{W} = m\mathbf{g}} \tag{1/3}$$

The *apparent weight* of a body as determined by a spring balance, calibrated to read the correct force and attached to the surface of the earth, will be slightly less than its true weight. The difference is due to the rotation of the earth. The ratio of the apparent weight to the apparent or relative acceleration due to gravity still gives the correct value of mass. The apparent weight and the relative acceleration due to gravity are, of course, the quantities which are measured in experiments conducted on the surface of the earth.

1/6 Dimensions

A given dimension such as length can be expressed in a number of different units such as meters, millimeters, or kilometers. Thus, a *dimension* is different from a *unit*. The *principle of dimensional homogeneity* states that all physical relations must be dimensionally homogeneous; that is, the dimensions of all terms in an equation must be the same. It is customary to use the symbols L, M, T, and F to stand for length, mass, time, and force, respectively. In SI units force is a derived quantity and from Eq. 1/1 has the dimensions of mass times acceleration or

$$F = ML/T^2$$

One important use of the dimensional homogeneity principle is to check the dimensional correctness of some derived physical relation. We can derive the following expression for the velocity v of a body of mass m which is moved from rest a horizontal distance x by a force F:

$$Fx = \tfrac{1}{2}mv^2$$

where the $\frac{1}{2}$ is a dimensionless coefficient resulting from integration. This equation is dimensionally correct because substitution of L, M, and T gives

$$[MLT^{-2}][L] = [M][LT^{-1}]^2$$

Dimensional homogeneity is a necessary condition for correctness of a physical relation, but it is not sufficient, since it is possible to construct

2 KINEMATICS OF PARTICLES

CHAPTER OUTLINE

2/1 INTRODUCTION

Kinematics is the branch of dynamics which describes the motion of bodies without reference to the forces which either cause the motion or are generated as a result of the motion. Kinematics is often described as the "geometry of motion." Some engineering applications of kinematics include the design of cams, gears, linkages, and other machine elements to control or produce certain desired motions, and the calculation of flight trajectories for aircraft, rockets, and spacecraft. A thorough working knowledge of kinematics is a prerequisite to kinetics, which is the study of the relationships between motion and the corresponding forces which cause or accompany the motion.

Particle Motion

We begin our study of kinematics by first discussing in this chapter the motions of points or particles. A particle is a body whose physical dimensions are so small compared with the radius of curvature of its path that we may treat the motion of the particle as that of a point. For example, the wingspan of a jet transport flying between Los Angeles and New York is of no consequence compared with the radius of curvature of

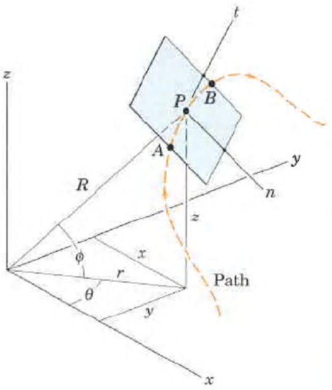


Figure 2/1

its flight path, and thus the treatment of the airplane as a particle or point is an acceptable approximation.

We can describe the motion of a particle in a number of ways, and the choice of the most convenient or appropriate way depends a great deal on experience and on how the data are given. Let us obtain an overview of the several methods developed in this chapter by referring to Fig. 2/1, which shows a particle P moving along some general path in space. If the particle is confined to a specified path, as with a bead sliding along a fixed wire, its motion is said to be *constrained*. If there are no physical guides, the motion is said to be *unconstrained*. A small rock tied to the end of a string and whirled in a circle undergoes constrained motion until the string breaks, after which instant its motion is unconstrained.

Choice of Coordinates

The position of particle P at any time t can be described by specifying its rectangular coordinates* x, y, z, its cylindrical coordinates r, θ, z, or its spherical coordinates R, θ, ϕ. The motion of P can also be described by measurements along the tangent t and normal n to the curve. The direction of n lies in the local plane of the curve.† These last two measurements are called *path variables*.

The motion of particles (or rigid bodies) can be described by using coordinates measured from fixed reference axes (*absolute-motion* analysis) or by using coordinates measured from moving reference axes (*relative-motion* analysis). Both descriptions will be developed and applied in the articles which follow.

With this conceptual picture of the description of particle motion in mind, we restrict our attention in the first part of this chapter to the case of *plane motion* where all movement occurs in or can be represented as occurring in a single plane. A large proportion of the motions of machines and structures in engineering can be represented as plane motion. Later, in Chapter 7, an introduction to three-dimensional motion is presented. We begin our discussion of plane motion with *rectilinear motion*, which is motion along a straight line, and follow it with a description of motion along a plane curve.

2/2 Rectilinear Motion

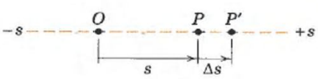


Figure 2/2

Consider a particle P moving along a straight line, Fig. 2/2. The position of P at any instant of time t can be specified by its distance s measured from some convenient reference point O fixed on the line. At time $t + \Delta t$ the particle has moved to P' and its coordinate becomes $s + \Delta s$. The change in the position coordinate during the interval Δt is called the *displacement* Δs of the particle. The displacement would be negative if the particle moved in the negative s-direction.

*Often called *Cartesian* coordinates, named after René Descartes (1596–1650), a French mathematician who was one of the inventors of analytic geometry.

†This plane is called the *osculating* plane, which comes from the Latin word *osculari* meaning "to kiss." The plane which contains P and the two points A and B, one on either side of P, becomes the osculating plane as the distances between the points approach zero.

Velocity and Acceleration

The average velocity of the particle during the interval Δt is the displacement divided by the time interval or $v_{av} = \Delta s/\Delta t$. As Δt becomes smaller and approaches zero in the limit, the average velocity approaches the *instantaneous velocity* of the particle, which is $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ or

$$\boxed{v = \frac{ds}{dt} = \dot{s}} \tag{2/1}$$

Thus, the velocity is the time rate of change of the position coordinate s. The velocity is positive or negative depending on whether the corresponding displacement is positive or negative.

The average acceleration of the particle during the interval Δt is the change in its velocity divided by the time interval or $a_{av} = \Delta v/\Delta t$. As Δt becomes smaller and approaches zero in the limit, the average acceleration approaches the *instantaneous acceleration* of the particle, which is $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ or

$$\boxed{a = \frac{dv}{dt} = \dot{v}} \quad \text{or} \quad \boxed{a = \frac{d^2s}{dt^2} = \ddot{s}} \tag{2/2}$$

The acceleration is positive or negative depending on whether the velocity is increasing or decreasing. Note that the acceleration would be positive if the particle had a negative velocity which was becoming less negative. If the particle is slowing down, the particle is said to be *decelerating*.

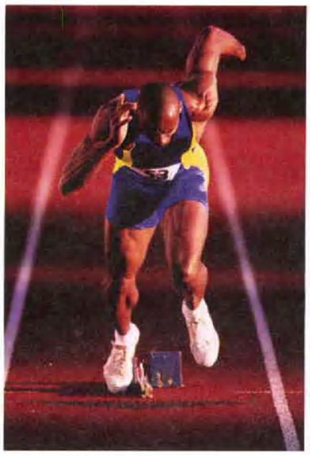
Velocity and acceleration are actually vector quantities, as we will see for curvilinear motion beginning with Art. 2/3. For rectilinear motion in the present article, where the direction of the motion is that of the given straight-line path, the sense of the vector along the path is described by a plus or minus sign. In our treatment of curvilinear motion, we will account for the changes in direction of the velocity and acceleration vectors as well as their changes in magnitude.

By eliminating the time dt between Eq. 2/1 and the first of Eqs. 2/2, we obtain a differential equation relating displacement, velocity, and acceleration.* This equation is

$$\boxed{v\,dv = a\,ds} \quad \text{or} \quad \boxed{\dot{s}\,d\dot{s} = \ddot{s}\,ds} \tag{2/3}$$

Equations 2/1, 2/2, and 2/3 are the differential equations for the rectilinear motion of a particle. Problems in rectilinear motion involving finite changes in the motion variables are solved by integration of these basic differential relations. The position coordinate s, the velocity v, and the acceleration a are algebraic quantities, so that their signs, positive or negative, must be carefully observed. Note that the positive directions for v and a are the same as the positive direction for s.

*Differential quantities can be multiplied and divided in exactly the same way as other algebraic quantities.



Jim Cummings/Taxi/Getty Images

This sprinter will undergo rectilinear acceleration until he reaches his terminal speed.

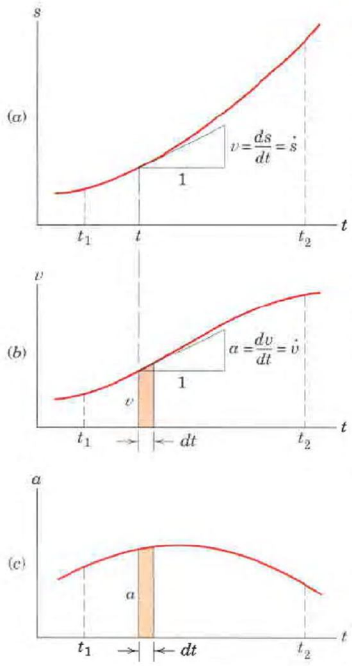


Figure 2/3

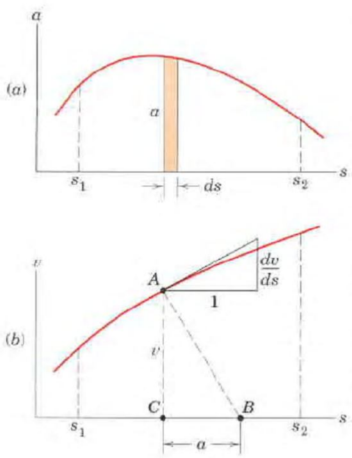


Figure 2/4

Graphical Interpretations

Interpretation of the differential equations governing rectilinear motion is considerably clarified by representing the relationships among s, v, a, and t graphically. Figure 2/3*a* is a schematic plot of the variation of s with t from time t_1 to time t_2 for some given rectilinear motion. By constructing the tangent to the curve at any time t, we obtain the slope, which is the velocity $v = ds/dt$. Thus, the velocity can be determined at all points on the curve and plotted against the corresponding time as shown in Fig. 2/3*b*. Similarly, the slope dv/dt of the v-t curve at any instant gives the acceleration at that instant, and the a-t curve can therefore be plotted as in Fig. 2/3*c*.

We now see from Fig. 2/3*b* that the area under the v-t curve during time dt is $v\,dt$, which from Eq. 2/1 is the displacement ds. Consequently, the net displacement of the particle during the interval from t_1 to t_2 is the corresponding area under the curve, which is

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt \qquad \text{or} \qquad s_2 - s_1 = (\text{area under } v\text{-}t \text{ curve})$$

Similarly, from Fig. 2/3*c* we see that the area under the a-t curve during time dt is $a\,dt$, which, from the first of Eqs. 2/2, is dv. Thus, the net change in velocity between t_1 and t_2 is the corresponding area under the curve, which is

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt \qquad \text{or} \qquad v_2 - v_1 = (\text{area under } a\text{-}t \text{ curve})$$

Note two additional graphical relations. When the acceleration a is plotted as a function of the position coordinate s, Fig. 2/4*a*, the area under the curve during a displacement ds is $a\,ds$, which, from Eq. 2/3, is $v\,dv = d(v^2/2)$. Thus, the net area under the curve between position coordinates s_1 and s_2 is

$$\int_{v_1}^{v_2} v\,dv = \int_{s_1}^{s_2} a\,ds \qquad \text{or} \qquad \tfrac{1}{2}(v_2{}^2 - v_1{}^2) = (\text{area under } a\text{-}s \text{ curve})$$

When the velocity v is plotted as a function of the position coordinate s, Fig. 2/4*b*, the slope of the curve at any point A is dv/ds. By constructing the normal AB to the curve at this point, we see from the similar triangles that $\overline{CB}/v = dv/ds$. Thus, from Eq. 2/3, $\overline{CB} = v(dv/ds) = a$, the acceleration. It is necessary that the velocity and position coordinate axes have the same numerical scales so that the acceleration read on the position coordinate scale in meters (or feet), say, will represent the actual acceleration in meters (or feet) per second squared.

The graphical representations described are useful not only in visualizing the relationships among the several motion quantities but also in obtaining approximate results by graphical integration or differentiation. The latter case occurs when a lack of knowledge of the mathematical relationship prevents its expression as an explicit mathematical function which can be integrated or differentiated. Experimental data and motions which involve discontinuous relationships between the variables are frequently analyzed graphically.

Analytical Integration

If the position coordinate s is known for all values of the time t, then successive mathematical or graphical differentiation with respect to t gives the velocity v and acceleration a. In many problems, however, the functional relationship between position coordinate and time is unknown, and we must determine it by successive integration from the acceleration. Acceleration is determined by the forces which act on moving bodies and is computed from the equations of kinetics discussed in subsequent chapters. Depending on the nature of the forces, the acceleration may be specified as a function of time, velocity, or position coordinate, or as a combined function of these quantities. The procedure for integrating the differential equation in each case is indicated as follows.

(a) Constant Acceleration. When a is constant, the first of Eqs. 2/2 and 2/3 can be integrated directly. For simplicity with $s = s_0$, $v = v_0$, and $t = 0$ designated at the beginning of the interval, then for a time interval t the integrated equations become

$$\int_{v_0}^{v} dv = a\int_0^t dt \qquad \text{or} \qquad v = v_0 + at$$

$$\int_{v_0}^{v} v\,dv = a\int_{s_0}^{s} ds \qquad \text{or} \qquad v^2 = v_0{}^2 + 2a(s - s_0)$$

Substitution of the integrated expression for v into Eq. 2/1 and integration with respect to t give

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + at)\,dt \qquad \text{or} \qquad s = s_0 + v_0 t + \tfrac{1}{2}at^2$$

These relations are necessarily restricted to the special case where the acceleration is constant. The integration limits depend on the initial and final conditions, which for a given problem may be different from those used here. It may be more convenient, for instance, to begin the integration at some specified time t_1 rather than at time $t = 0$.

Caution: The foregoing equations have been integrated for constant acceleration only. A common mistake is to use these equations for problems involving variable acceleration, where they do not apply.

(b) Acceleration Given as a Function of Time, $a = f(t)$. Substitution of the function into the first of Eqs. 2/2 gives $f(t) = dv/dt$. Multiplying by dt separates the variables and permits integration. Thus,

$$\int_{v_0}^{v} dv = \int_0^t f(t)\,dt \qquad \text{or} \qquad v = v_0 + \int_0^t f(t)\,dt$$

From this integrated expression for v as a function of t, the position coordinate s is obtained by integrating Eq. 2/1, which, in form, would be

$$\int_{s_0}^{s} ds = \int_0^t v\,dt \qquad \text{or} \qquad s = s_0 + \int_0^t v\,dt$$

If the indefinite integral is employed, the end conditions are used to establish the constants of integration. The results are identical with those obtained by using the definite integral.

If desired, the displacement s can be obtained by a direct solution of the second-order differential equation $\ddot{s} = f(t)$ obtained by substitution of $f(t)$ into the second of Eqs. 2/2.

(c) Acceleration Given as a Function of Velocity, $a = f(v)$. Substitution of the function into the first of Eqs. 2/2 gives $f(v) = dv/dt$, which permits separating the variables and integrating. Thus,

$$t = \int_0^t dt = \int_{v_0}^{v} \frac{dv}{f(v)}$$

This result gives t as a function of v. Then it would be necessary to solve for v as a function of t so that Eq. 2/1 can be integrated to obtain the position coordinate s as a function of t.

Another approach is to substitute the function $a = f(v)$ into the first of Eqs. 2/3, giving $v\,dv = f(v)\,ds$. The variables can now be separated and the equation integrated in the form

$$\int_{v_0}^{v} \frac{v\,dv}{f(v)} = \int_{s_0}^{s} ds \qquad \text{or} \qquad s = s_0 + \int_{v_0}^{v} \frac{v\,dv}{f(v)}$$

Note that this equation gives s in terms of v without explicit reference to t.

(d) Acceleration Given as a Function of Displacement, $a = f(s)$. Substituting the function into Eq. 2/3 and integrating give the form

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} f(s)\,ds \qquad \text{or} \qquad v^2 = {v_0}^2 + 2\int_{s_0}^{s} f(s)\,ds$$

Next we solve for v to give $v = g(s)$, a function of s. Now we can substitute ds/dt for v, separate variables, and integrate in the form

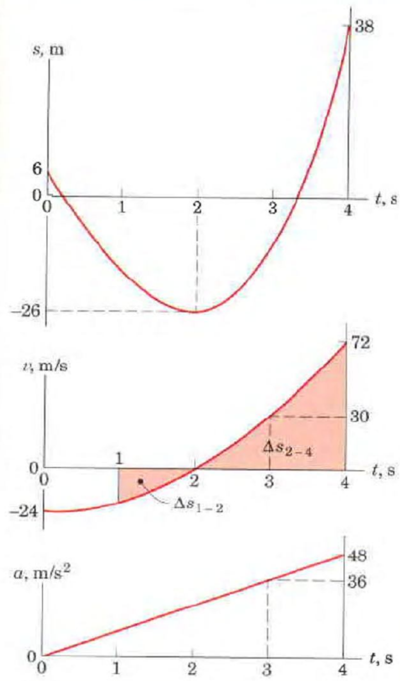
$$\int_{s_0}^{s} \frac{ds}{g(s)} = \int_0^t dt \qquad \text{or} \qquad t = \int_{s_0}^{s} \frac{ds}{g(s)}$$

which gives t as a function of s. Finally, we can rearrange to obtain s as a function of t.

In each of the foregoing cases when the acceleration varies according to some functional relationship, the possibility of solving the equations by direct mathematical integration will depend on the form of the function. In cases where the integration is excessively awkward or difficult, integration by graphical, numerical, or computer methods can be utilized.

Sample Problem 2/1

The position coordinate of a particle which is confined to move along a straight line is given by $s = 2t^3 - 24t + 6$, where s is measured in meters from a convenient origin and t is in seconds. Determine (*a*) the time required for the particle to reach a velocity of 72 m/s from its initial condition at $t = 0$, (*b*) the acceleration of the particle when $v = 30$ m/s, and (*c*) the net displacement of the particle during the interval from $t = 1$ s to $t = 4$ s.



Solution. The velocity and acceleration are obtained by successive differentiation of s with respect to the time. Thus,

$[v = \dot{s}]$ $\qquad v = 6t^2 - 24$ m/s

$[a = \dot{v}]$ $\qquad a = 12t$ m/s^2

(a) Substituting $v = 72$ m/s into the expression for v gives us $72 = 6t^2 - 24$, from which $t = \pm 4$ s. The negative root describes a mathematical solution for t before ① the initiation of motion, so this root is of no physical interest. Thus, the desired result is

$$t = 4 \text{ s} \qquad \textit{Ans.}$$

(b) Substituting $v = 30$ m/s into the expression for v gives $30 = 6t^2 - 24$, from which the positive root is $t = 3$ s, and the corresponding acceleration is

$$a = 12(3) = 36 \text{ m/s}^2 \qquad \textit{Ans.}$$

(c) The net displacement during the specified interval is

$$\Delta s = s_4 - s_1 \qquad \text{or}$$

$$\Delta s = [2(4^3) - 24(4) + 6] - [2(1^3) - 24(1) + 6]$$

$$= 54 \text{ m} \qquad \textit{Ans.}$$

② which represents the net advancement of the particle along the s-axis from the position it occupied at $t = 1$ s to its position at $t = 4$ s.

To help visualize the motion, the values of s, v, and a are plotted against the time t as shown. Because the area under the v-t curve represents displacement, ③ we see that the net displacement from $t = 1$ s to $t = 4$ s is the positive area Δs_{2-4} less the negative area Δs_{1-2}.

Helpful Hints

① Be alert to the proper choice of sign when taking a square root. When the situation calls for only one answer, the positive root is not always the one you may need.

② Note carefully the distinction between italic s for the position coordinate and the vertical s for seconds.

③ Note from the graphs that the values for v are the slopes ($\dot{s}$) of the s-t curve and that the values for a are the slopes ($\dot{v}$) of the v-t curve. *Suggestion:* Integrate $v\,dt$ for each of the two intervals and check the answer for Δs. Show that the total distance traveled during the interval $t = 1$ s to $t = 4$ s is 74 m.

Sample Problem 2/2

A particle moves along the x-axis with an initial velocity $v_x = 50$ ft/sec at the origin when $t = 0$. For the first 4 seconds it has no acceleration, and thereafter it is acted on by a retarding force which gives it a constant acceleration $a_x = -10$ ft/sec^2. Calculate the velocity and the x-coordinate of the particle for the conditions of $t = 8$ sec and $t = 12$ sec and find the maximum positive x-coordinate reached by the particle. ①

Helpful Hints

① Learn to be flexible with symbols. The position coordinate x is just as valid as s.

Solution. The velocity of the particle after $t = 4$ sec is computed from

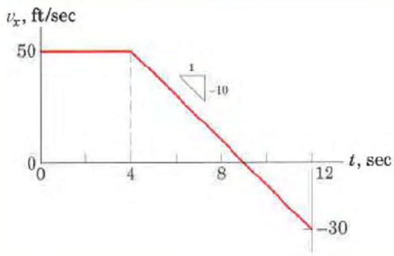
② $$\left[\int dv = \int a\, dt\right] \qquad \int_{50}^{v_x} dv_x = -10 \int_4^t dt \qquad v_x = 90 - 10t \text{ ft/sec}$$

② Note that we integrate to a general time t and then substitute specific values.

and is plotted as shown. At the specified times, the velocities are

$$t = 8 \text{ sec}, \qquad v_x = 90 - 10(8) = 10 \text{ ft/sec}$$

$$t = 12 \text{ sec}, \qquad v_x = 90 - 10(12) = -30 \text{ ft/sec} \qquad \textit{Ans.}$$



The x-coordinate of the particle at any time greater than 4 seconds is the distance traveled during the first 4 seconds plus the distance traveled after the discontinuity in acceleration occurred. Thus,

$$\left[\int ds = \int v\, dt\right] \qquad x = 50(4) + \int_4^t (90 - 10t)\, dt = -5t^2 + 90t - 80 \text{ ft}$$

For the two specified times,

$$t = 8 \text{ sec}, \qquad x = -5(8^2) + 90(8) - 80 = 320 \text{ ft}$$

$$t = 12 \text{ sec}, \qquad x = -5(12^2) + 90(12) - 80 = 280 \text{ ft} \qquad \textit{Ans.}$$

The x-coordinate for $t = 12$ sec is less than that for $t = 8$ sec since the motion is in the negative x-direction after $t = 9$ sec. The maximum positive x-coordinate is, then, the value of x for $t = 9$ sec which is

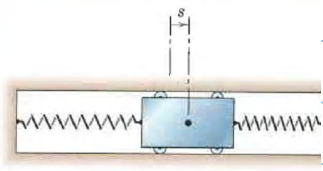
$$x_{max} = -5(9^2) + 90(9) - 80 = 325 \text{ ft} \qquad \textit{Ans.}$$

③ These displacements are seen to be the net positive areas under the v-t graph up to the values of t in question.

③ Show that the total distance traveled by the particle in the 12 sec is 370 ft.

Sample Problem 2/3

The spring-mounted slider moves in the horizontal guide with negligible friction and has a velocity v_0 in the s-direction as it crosses the mid-position where $s = 0$ and $t = 0$. The two springs together exert a retarding force to the motion of the slider, which gives it an acceleration proportional to the displacement but oppositely directed and equal to $a = -k^2 s$, where k is constant. (The constant is arbitrarily squared for later convenience in the form of the expressions.) Determine the expressions for the displacement s and velocity v as functions of the time t.



Solution I. Since the acceleration is specified in terms of the displacement, the differential relation $v\,dv = a\,ds$ may be integrated. Thus,

① $$\int v\,dv = \int -k^2 s\,ds + C_1 \text{ a constant,} \quad \text{or} \quad \frac{v^2}{2} = -\frac{k^2 s^2}{2} + C_1$$

When $s = 0$, $v = v_0$, so that $C_1 = v_0{}^2/2$, and the velocity becomes

$$v = +\sqrt{v_0{}^2 - k^2 s^2}$$

The plus sign of the radical is taken when v is positive (in the plus s-direction). This last expression may be integrated by substituting $v = ds/dt$. Thus,

② $$\int \frac{ds}{\sqrt{v_0{}^2 - k^2 s^2}} = \int dt + C_2 \text{ a constant,} \quad \text{or} \quad \frac{1}{k}\sin^{-1}\frac{ks}{v_0} = t + C_2$$

With the requirement of $t = 0$ when $s = 0$, the constant of integration becomes $C_2 = 0$, and we may solve the equation for s so that

$$s = \frac{v_0}{k}\sin kt \qquad \textit{Ans.}$$

The velocity is $v = \dot{s}$, which gives

$$v = v_0 \cos kt \qquad \textit{Ans.}$$

Solution II. Since $a = \ddot{s}$, the given relation may be written at once as

$$\ddot{s} + k^2 s = 0$$

This is an ordinary linear differential equation of second order for which the solution is well known and is

$$s = A \sin Kt + B \cos Kt$$

where A, B, and K are constants. Substitution of this expression into the differential equation shows that it satisfies the equation, provided that $K = k$. The velocity is $v = \dot{s}$, which becomes

$$v = Ak \cos kt - Bk \sin kt$$

The initial condition $v = v_0$ when $t = 0$ requires that $A = v_0/k$, and the condition $s = 0$ when $t = 0$ gives $B = 0$. Thus, the solution is

③ $$s = \frac{v_0}{k}\sin kt \quad \text{and} \quad v = v_0 \cos kt \qquad \textit{Ans.}$$

Helpful Hints

① We have used an indefinite integral here and evaluated the constant of integration. For practice, obtain the same results by using the definite integral with the appropriate limits.

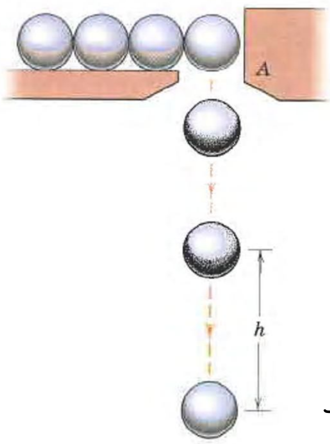
② Again try the definite integral here as above.

$$\int \frac{ds}{\sqrt{v_0^2 - k^2 s^2}} =$$

③ This motion is called *simple harmonic motion* and is characteristic of all oscillations where the restoring force, and hence the acceleration, is proportional to the displacement but opposite in sign.

2/23 Small steel balls fall from rest through the opening at A at the steady rate of two per second. Find the vertical separation h of two consecutive balls when the lower one has dropped 3 meters. Neglect air resistance.

Ans. $h = 2.61$ m



$$y_1 = -\frac{1}{2} g t^2 + \cancel{v_0 t} \quad = 0$$

3 m y g

$$-3 = -\frac{1}{2} \times 9{,}81 \times t_1^2$$

$$t_1 = \sqrt{\frac{6}{9.81}}$$

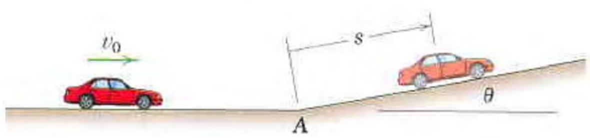
$$t_2 = t_1 - 0{,}5$$

$$y_2 = -\frac{1}{2} \times 9{,}81 \times t_2^2$$

$$h = y_1 - y_2$$

2/29 The car is traveling at a constant speed $v_0 = 100$ km/h on the level portion of the road. When the 6-percent ($\tan\theta = 6/100$) incline is encountered, the driver does not change the throttle setting and consequently the car decelerates at the constant rate $g \sin\theta$. Determine the speed of the car (a) 10 seconds after passing point A and (b) when $s = 100$ m.

Ans. (a) $v = 21.9$ m/s, (b) $v = 25.6$ m/s



2/40 The cone falling with a speed v_0 strikes and penetrates the block of packing material. The acceleration of the cone after impact is $a = g - cy^2$, where c is a positive constant and y is the penetration distance. If the maximum penetration depth is observed to be y_m, determine the constant c.