MATHEMATICAL HANDBOOK

of FORMULAS and TABLES

MURRAY R. SPIEGEL

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MATHEMATICAL HANDBOOK

of

Formulas and Tables

BY

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Preface

The purpose of this handbook is to supply a collection of mathematical formulas and tables which will prove to be valuable to students and research workers in the fields of mathematics, physics, engineering and other sciences. To accomplish this, care has been taken to include those formulas and tables which are most likely to be needed in practice rather than highly specialized results which are rarely used. Every effort has been made to present results concisely as well as precisely so that they may be referred to with a maximum of ease as well as confidence.

Topics covered range from elementary to advanced. Elementary topics include those from algebra, geometry, trigonometry, analytic geometry and calculus. Advanced topics include those from differential equations, vector analysis, Fourier series, gamma and beta functions, Bessel and Legendre functions, Fourier and Laplace transforms, elliptic functions and various other special functions of importance. This wide coverage of topics has been adopted so as to provide within a single volume most of the important mathematical results needed by the student or research worker regardless of his particular field of interest or level of attainment.

The book is divided into two main parts. Part I presents mathematical formulas together with other material, such as definitions, theorems, graphs, diagrams, etc., essential for proper understanding and application of the formulas. Included in this first part are extensive tables of integrals and Laplace transforms which should be extremely useful to the student and research worker. Part II presents numerical tables such as the values of elementary functions (trigonometric, logarithmic, exponential, hyperbolic, etc.) as well as advanced functions (Bessel, Legendre, elliptic, etc.). In order to eliminate confusion, especially to the beginner in mathematics, the numerical tables for each function are separated. Thus, for example, the sine and cosine functions for angles in degrees and minutes are given in separate tables rather than in one table so that there is no need to be concerned about the possibility of error due to looking in the wrong column or row.

I wish to thank the various authors and publishers who gave me permission to adapt data from their books for use in several tables of this handbook. Appropriate references to such sources are given next to the corresponding tables. In particular I am indebted to the Literary Executor of the late Sir Ronald A. Fisher, F.R.S., to Dr. Frank Yates, F.R.S., and to Oliver and Boyd Ltd., Edinburgh, for permission to use data from Table III of their book *Statistical Tables for Biological, Agricultural and Medical Research.*

I also wish to express my gratitude to Nicola Monti, Henry Hayden and Jack Margolin for their excellent editorial cooperation.

M. R. SPIEGEL

Rensselaer Polytechnic Institute
September, 1968

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Part I — FORMULAS

Part II TABLES

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Part I

FORMULAS

THE GREEK ALPHABET

Greek name	Greek letter	
	Lower case	Capital
Alpha	α	A
Beta	β	B
Gamma	γ	Γ
Delta	δ	Δ
Epsilon	ϵ	E
Zeta	ζ	Z
Eta	η	H
Theta	θ	Θ
Iota	ι	I
Kappa	κ	K
Lambda	λ	Λ
Mu	μ	M

Greek name	Greek letter	
	Lower case	Capital
Nu	ν	N
Xi	ξ	Ξ
Omicron	o	O
Pi	π	Π
Rho	ρ	P
Sigma	σ	Σ
Tau	τ	T
Upsilon	υ	Υ
Phi	ϕ	Φ
Chi	χ	X
Psi	ψ	Ψ
Omega	ω	Ω

1 SPECIAL CONSTANTS

1.1 $\pi = 3.14159\ 26535\ 89793\ 23846\ 2643\ldots$

1.2 $e = 2.71828\ 18284\ 59045\ 23536\ 0287\ldots = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
$=$ natural base of logarithms

1.3 $\sqrt{2} = 1.41421\ 35623\ 73095\ 0488\ldots$

1.4 $\sqrt{3} = 1.73205\ 08075\ 68877\ 2935\ldots$

1.5 $\sqrt{5} = 2.23606\ 79774\ 99789\ 6964\ldots$

1.6 $\sqrt[3]{2} = 1.25992\ 1050\ldots$

1.7 $\sqrt[3]{3} = 1.44224\ 9570\ldots$

1.8 $\sqrt[5]{2} = 1.14869\ 8355\ldots$

1.9 $\sqrt[5]{3} = 1.24573\ 0940\ldots$

1.10 $e^\pi = 23.14069\ 26327\ 79269\ 006\ldots$

1.11 $\pi^e = 22.45915\ 77183\ 61045\ 47342\ 715\ldots$

1.12 $e^e = 15.15426\ 22414\ 79264\ 190\ldots$

1.13 $\log_{10} 2 = 0.30102\ 99956\ 63981\ 19521\ 37389\ldots$

1.14 $\log_{10} 3 = 0.47712\ 12547\ 19662\ 43729\ 50279\ldots$

1.15 $\log_{10} e = 0.43429\ 44819\ 03251\ 82765\ldots$

1.16 $\log_{10} \pi = 0.49714\ 98726\ 94133\ 85435\ 12683\ldots$

1.17 $\log_e 10 = \ln 10 = 2.30258\ 50929\ 94045\ 68401\ 7991\ldots$

1.18 $\log_e 2 = \ln 2 = 0.69314\ 71805\ 59945\ 30941\ 7232\ldots$

1.19 $\log_e 3 = \ln 3 = 1.09861\ 22886\ 68109\ 69139\ 5245\ldots$

1.20 $\gamma = 0.57721\ 56649\ 01532\ 86060\ 6512\ldots =$ *Euler's constant*
$= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)$

1.21 $e^\gamma = 1.78107\ 24179\ 90197\ 9852\ldots$ [see 1.20]

1.22 $\sqrt{e} = 1.64872\ 12707\ 00128\ 1468\ldots$

1.23 $\sqrt{\pi} = \Gamma(\frac{1}{2}) = 1.77245\ 38509\ 05516\ 02729\ 8167\ldots$
where Γ is the *gamma function* [see pages 101-102].

1.24 $\Gamma(\frac{1}{3}) = 2.67893\ 85347\ 07748\ldots$

1.25 $\Gamma(\frac{1}{4}) = 3.62560\ 99082\ 21908\ldots$

1.26 $1 \text{ radian} = 180°/\pi = 57.29577\ 95130\ 8232\ldots°$

1.27 $1° = \pi/180 \text{ radians} = 0.01745\ 32925\ 19943\ 29576\ 92\ldots$ radians

2 SPECIAL PRODUCTS and FACTORS

2.1 $(x+y)^2 = x^2 + 2xy + y^2$

2.2 $(x-y)^2 = x^2 - 2xy + y^2$

2.3 $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2.4 $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

2.5 $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

2.6 $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

2.7 $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

2.8 $(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

2.9 $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

2.10 $(x-y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

The results 2.1 to 2.10 above are special cases of the *binomial formula* [see page 3].

2.11 $x^2 - y^2 = (x-y)(x+y)$

2.12 $x^3 - y^3 = (x-y)(x^2+xy+y^2)$

2.13 $x^3 + y^3 = (x+y)(x^2-xy+y^2)$

2.14 $x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$

2.15 $x^5 - y^5 = (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$

2.16 $x^5 + y^5 = (x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$

2.17 $x^6 - y^6 = (x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$

2.18 $x^4 + x^2y^2 + y^4 = (x^2+xy+y^2)(x^2-xy+y^2)$

2.19 $x^4 + 4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$

Some generalizations of the above are given by the following results where n is a positive integer.

2.20
$$\begin{aligned} x^{2n+1} - y^{2n+1} &= (x-y)(x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \cdots + y^{2n}) \\ &= (x-y)\left(x^2 - 2xy\cos\frac{2\pi}{2n+1} + y^2\right)\left(x^2 - 2xy\cos\frac{4\pi}{2n+1} + y^2\right) \\ &\qquad \cdots \left(x^2 - 2xy\cos\frac{2n\pi}{2n+1} + y^2\right) \end{aligned}$$

2.21
$$\begin{aligned} x^{2n+1} + y^{2n+1} &= (x+y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots + y^{2n}) \\ &= (x+y)\left(x^2 + 2xy\cos\frac{2\pi}{2n+1} + y^2\right)\left(x^2 + 2xy\cos\frac{4\pi}{2n+1} + y^2\right) \\ &\qquad \cdots \left(x^2 + 2xy\cos\frac{2n\pi}{2n+1} + y^2\right) \end{aligned}$$

2.22
$$\begin{aligned} x^{2n} - y^{2n} &= (x-y)(x+y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots) \\ &= (x-y)(x+y)\left(x^2 - 2xy\cos\frac{\pi}{n} + y^2\right)\left(x^2 - 2xy\cos\frac{2\pi}{n} + y^2\right) \\ &\qquad \cdots \left(x^2 - 2xy\cos\frac{(n-1)\pi}{n} + y^2\right) \end{aligned}$$

2.23
$$\begin{aligned} x^{2n} + y^{2n} &= \left(x^2 + 2xy\cos\frac{\pi}{2n} + y^2\right)\left(x^2 + 2xy\cos\frac{3\pi}{2n} + y^2\right) \\ &\qquad \cdots \left(x^2 + 2xy\cos\frac{(2n-1)\pi}{2n} + y^2\right) \end{aligned}$$

3 The BINOMIAL FORMULA and BINOMIAL COEFFICIENTS

FACTORIAL n

If $n = 1, 2, 3, \ldots$ *factorial* n or n *factorial* is defined as

3.1 $$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

We also define *zero factorial* as

3.2 $$0! = 1$$

BINOMIAL FORMULA FOR POSITIVE INTEGRAL n

If $n = 1, 2, 3, \ldots$ then

3.3 $$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n$$

This is called the *binomial formula*. It can be extended to other values of n and then is an infinite series [see *Binomial Series*, page 110].

BINOMIAL COEFFICIENTS

The result 3.3 can also be written

3.4 $$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}y^n$$

where the coefficients, called *binomial coefficients*, are given by

3.5 $$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

PROPERTIES OF BINOMIAL COEFFICIENTS

3.6 $$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

This leads to *Pascal's triangle* [see page 236].

3.7 $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

3.8 $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots (-1)^n \binom{n}{n} = 0$$

3.9 $$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+m}{n} = \binom{n+m+1}{n+1}$$

3.10 $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$$

3.11 $$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$$

3.12 $$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

3.13 $$\binom{m}{0}\binom{n}{p} + \binom{m}{1}\binom{n}{p-1} + \cdots + \binom{m}{p}\binom{n}{0} = \binom{m+n}{p}$$

3.14 $$(1)\binom{n}{1} + (2)\binom{n}{2} + (3)\binom{n}{3} + \cdots + (n)\binom{n}{n} = n2^{n-1}$$

3.15 $$(1)\binom{n}{1} - (2)\binom{n}{2} + (3)\binom{n}{3} - \cdots (-1)^{n+1}(n)\binom{n}{n} = 0$$

MULTINOMIAL FORMULA

3.16 $$(x_1 + x_2 + \cdots + x_p)^n = \sum \frac{n!}{n_1!\, n_2! \cdots n_p!} x_1^{n_1} x_2^{n_2} \cdots x_p^{n_p}$$

where the sum, denoted by Σ, is taken over all nonnegative integers $n_1, n_2, \ldots, n_p$ for which $n_1 + n_2 + \cdots + n_p = n$.

4 GEOMETRIC FORMULAS

RECTANGLE OF LENGTH b AND WIDTH a

4.1 Area $= ab$

4.2 Perimeter $= 2a + 2b$

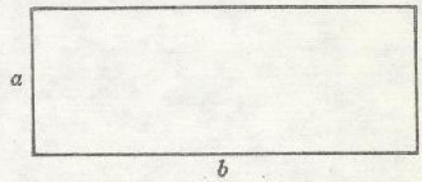

Fig. 4-1

PARALLELOGRAM OF ALTITUDE h AND BASE b

4.3 Area $= bh = ab \sin\theta$

4.4 Perimeter $= 2a + 2b$

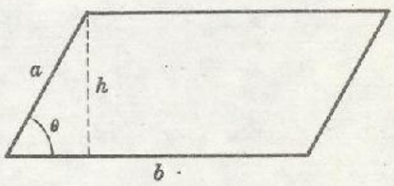

Fig. 4-2

TRIANGLE OF ALTITUDE h AND BASE b

4.5 Area $= \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

4.6 Perimeter $= a + b + c$

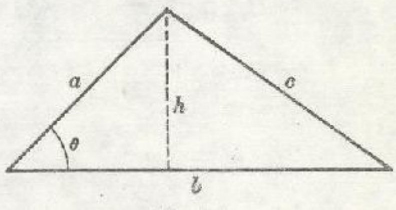

Fig. 4-3

TRAPEZOID OF ALTITUDE h AND PARALLEL SIDES a AND b

4.7 Area $= \frac{1}{2}h(a+b)$

4.8 Perimeter $= a + b + h\left(\frac{1}{\sin\theta} + \frac{1}{\sin\phi}\right)$

$$= a + b + h(\csc\theta + \csc\phi)$$

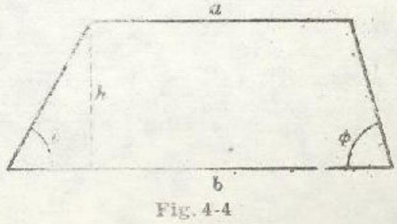

Fig. 4-4

REGULAR POLYGON OF n SIDES EACH OF LENGTH b

4.9 Area $= \frac{1}{4}nb^2 \cot\frac{\pi}{n} = \frac{1}{4}nb^2 \frac{\cos(\pi/n)}{\sin(\pi/n)}$

4.10 Perimeter $= nb$

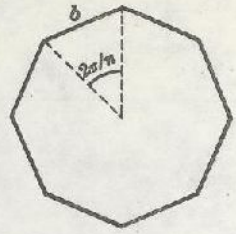

Fig. 4-5

CIRCLE OF RADIUS r

4.11 Area $= \pi r^2$

4.12 Perimeter $= 2\pi r$

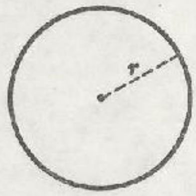

Fig. 4-6

SECTOR OF CIRCLE OF RADIUS r

4.13 Area $= \frac{1}{2}r^2\theta$ [θ in radians]

4.14 Arc length $s = r\theta$

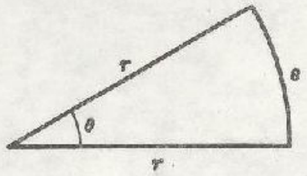

Fig. 4-7

RADIUS OF CIRCLE INSCRIBED IN A TRIANGLE OF SIDES a, b, c

4.15
$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

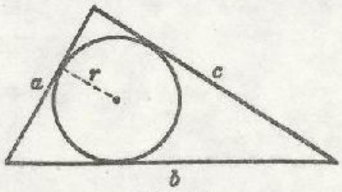

Fig. 4-8

RADIUS OF CIRCLE CIRCUMSCRIBING A TRIANGLE OF SIDES a, b, c

4.16
$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

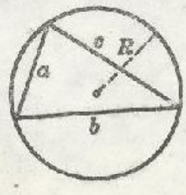

Fig. 4-9

REGULAR POLYGON OF n SIDES INSCRIBED IN CIRCLE OF RADIUS r

4.17 Area $= \frac{1}{2}nr^2 \sin\frac{2\pi}{n} = \frac{1}{2}nr^2 \sin\frac{360^\circ}{n}$

4.18 Perimeter $= 2nr \sin\frac{\pi}{n} = 2nr \sin\frac{180^\circ}{n}$

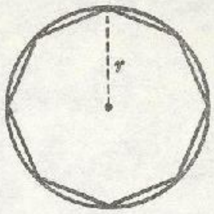

Fig. 4-10

REGULAR POLYGON OF n SIDES CIRCUMSCRIBING A CIRCLE OF RADIUS r

4.19 Area $= nr^2 \tan\frac{\pi}{n} = nr^2 \tan\frac{180^\circ}{n}$

4.20 Perimeter $= 2nr \tan\frac{\pi}{n} = 2nr \tan\frac{180^\circ}{n}$

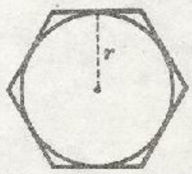

Fig. 4-11

SEGMENT OF CIRCLE OF RADIUS r

4.21 Area of shaded part $= \frac{1}{2}r^2(\theta - \sin\theta)$

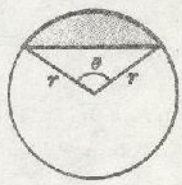

Fig. 4-12

ELLIPSE OF SEMI-MAJOR AXIS a AND SEMI-MINOR AXIS b

4.22 Area $= \pi ab$

4.23 Perimeter $= 4a\int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$

$= 2\pi\sqrt{\frac{1}{2}(a^2 + b^2)}$ [approximately]

where $k = \sqrt{a^2 - b^2}/a$. See page 254 for numerical tables.

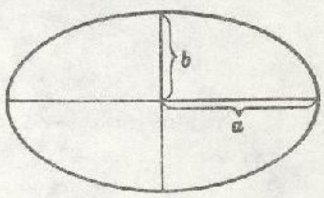

Fig. 4-13

SEGMENT OF A PARABOLA

4.24 Area $= \frac{2}{3}ab$

4.25 Arc length $ABC = \frac{1}{2}\sqrt{b^2 + 16a^2} + \frac{b^2}{8a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$

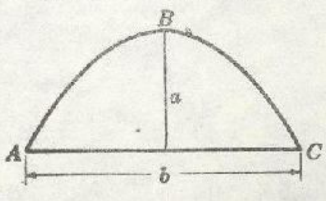

Fig. 4-14

RECTANGULAR PARALLELEPIPED OF LENGTH a, HEIGHT l, WIDTH c

4.26 Volume $= abc$

4.27 Surface area $= 2(ab + ac + bc)$

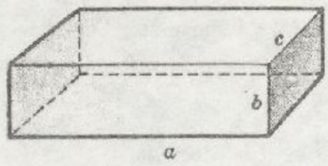

Fig. 4-15

PARALLELEPIPED OF CROSS-SECTIONAL AREA A AND HEIGHT h

4.28 Volume $= Ah = abc \sin\theta$

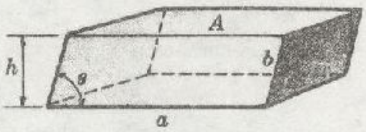

Fig. 4-16

SPHERE OF RADIUS r

4.29 Volume $= \frac{4}{3}\pi r^3$

4.30 Surface area $= 4\pi r^2$

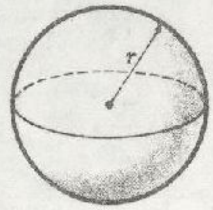

Fig. 4-17

RIGHT CIRCULAR CYLINDER OF RADIUS r AND HEIGHT h

4.31 Volume $= \pi r^2 h$

4.32 Lateral surface area $= 2\pi rh$

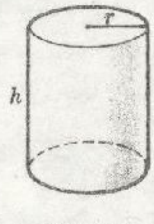

Fig. 4-18

CIRCULAR CYLINDER OF RADIUS r AND SLANT HEIGHT l

4.33 Volume $= \pi r^2 h = \pi r^2 l \sin\theta$

4.34 Lateral surface area $= 2\pi rl = \dfrac{2\pi rh}{\sin\theta} = 2\pi rh \csc\theta$

Fig. 4-19

CYLINDER OF CROSS-SECTIONAL AREA A AND SLANT HEIGHT l

4.35 Volume $= Ah = Al \sin\theta$

4.36 Lateral surface area $= pl = \dfrac{ph}{\sin\theta} = ph \csc\theta$

Note that formulas 4.31 to 4.34 are special cases.

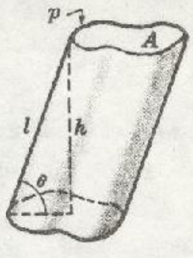

Fig. 4-20

RIGHT CIRCULAR CONE OF RADIUS r AND HEIGHT h

4.37 Volume $= \frac{1}{3}\pi r^2 h$

4.38 Lateral surface area $= \pi r\sqrt{r^2+h^2} = \pi r l$

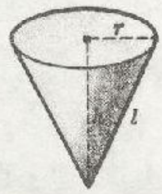

Fig. 4-21

PYRAMID OF BASE AREA A AND HEIGHT h

4.39 Volume $= \frac{1}{3}Ah$

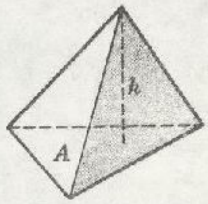

Fig. 4-22

SPHERICAL CAP OF RADIUS r AND HEIGHT h

4.40 Volume (shaded in figure) $= \frac{1}{3}\pi h^2(3r-h)$

4.41 Surface area $= 2\pi r h$

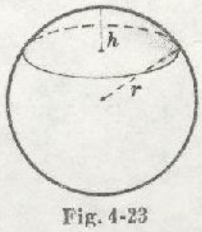

Fig. 4-23

FRUSTRUM OF RIGHT CIRCULAR CONE OF RADII a, b AND HEIGHT h

4.42 Volume $= \frac{1}{3}\pi h(a^2+ab+b^2)$

4.43 Lateral surface area $= \pi(a+b)\sqrt{h^2+(b-a)^2}$

$= \pi(a+b)l$

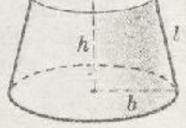

Fig. 4-24

SPHERICAL TRIANGLE OF ANGLES A, B, C ON SPHERE OF RADIUS r

4.44 Area of triangle $ABC = (A+B+C-\pi)r^2$

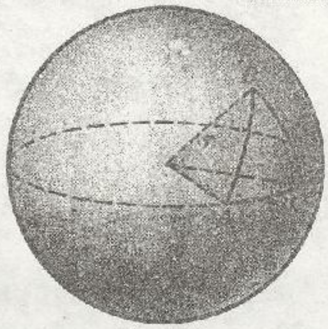

Fig. 4-25

TORUS OF INNER RADIUS a AND OUTER RADIUS b

4.45 Volume $= \frac{1}{4}\pi^2(a+b)(b-a)^2$

4.46 Surface area $= \pi^2(b^2-a^2)$

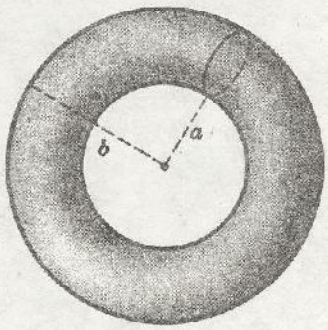

Fig. 4-26

ELLIPSOID OF SEMI-AXES a, b, c

4.47 Volume $= \frac{4}{3}\pi abc$

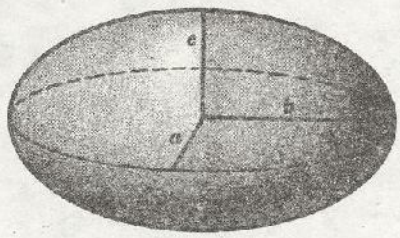

Fig. 4-27

PARABOLOID OF REVOLUTION

4.48 Volume $= \frac{1}{2}\pi b^2 a$

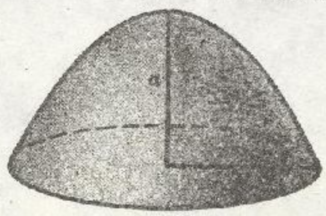

Fig. 4-28

5 TRIGONOMETRIC FUNCTIONS

DEFINITION OF TRIGONOMETRIC FUNCTIONS FOR A RIGHT TRIANGLE

Triangle ABC has a right angle (90°) at C and sides of length a, b, c. The trigonometric functions of angle A are defined as follows.

5.1 $$\textit{sine}\text{ of } A = \sin A = \frac{a}{c} = \frac{\text{opposite}}{\text{hypotenuse}}$$

5.2 $$\textit{cosine}\text{ of } A = \cos A = \frac{b}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

5.3 $$\textit{tangent}\text{ of } A = \tan A = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}}$$

5.4 $$\textit{cotangent}\text{ of } A = \cot A = \frac{b}{a} = \frac{\text{adjacent}}{\text{opposite}}$$

5.5 $$\textit{secant}\text{ of } A = \sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

5.6 $$\textit{cosecant}\text{ of } A = \csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite}}$$

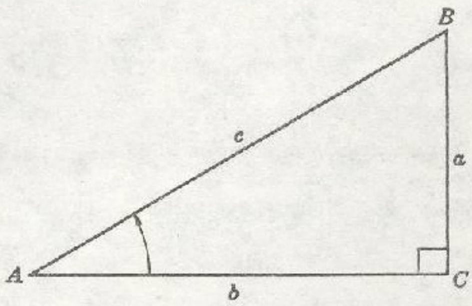

Fig. 5-1

EXTENSIONS TO ANGLES WHICH MAY BE GREATER THAN 90°

Consider an xy coordinate system [see Fig. 5-2 and 5-3 below]. A point P in the xy plane has coordinates (x, y) where x is considered as positive along OX and negative along OX' while y is positive along OY and negative along OY'. The distance from origin O to point P is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle A described *counterclockwise* from OX is considered *positive*. If it is described *clockwise* from OX it is considered *negative*. We call $X'OX$ and $Y'OY$ the x and y axis respectively.

The various quadrants are denoted by I, II, III and IV called the first, second, third and fourth quadrants respectively. In Fig. 5-2, for example, angle A is in the second quadrant while in Fig. 5-3 angle A is in the third quadrant.

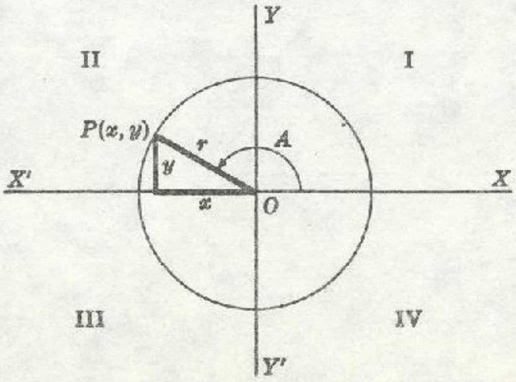

Fig. 5-2

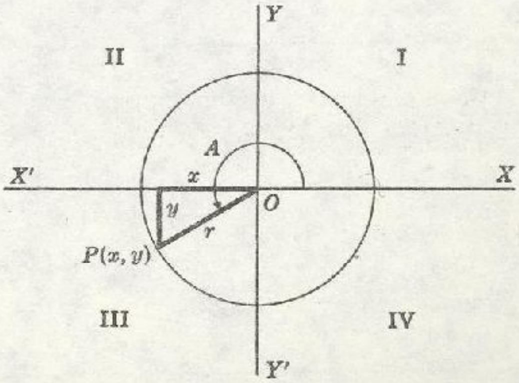

Fig. 5-3

For an angle A in any quadrant the trigonometric functions of A are defined as follows.

5.7 $$\sin A = y/r$$

5.8 $$\cos A = x/r$$

5.9 $$\tan A = y/x$$

5.10 $$\cot A = x/y$$

5.11 $$\sec A = r/x$$

5.12 $$\csc A = r/y$$

RELATIONSHIP BETWEEN DEGREES AND RADIANS

A *radian* is that angle θ subtended at center O of a circle by an arc MN equal to the radius r.

Since 2π radians $= 360°$ we have

5.13 $$1 \text{ radian} = 180°/\pi = 57.29577\ 95130\ 8232\ldots°$$

5.14 $$1° = \pi/180 \text{ radians} = 0.01745\ 32925\ 19943\ 29576\ 92\ldots \text{radians}$$

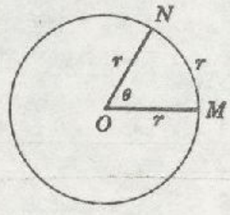

Fig. 5-4

RELATIONSHIPS AMONG TRIGONOMETRIC FUNCTIONS

5.15 $$\tan A = \frac{\sin A}{\cos A}$$

5.16 $$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

5.17 $$\sec A = \frac{1}{\cos A}$$

5.18 $$\csc A = \frac{1}{\sin A}$$

5.19 $$\sin^2 A + \cos^2 A = 1$$

5.20 $$\sec^2 A - \tan^2 A = 1$$

5.21 $$\csc^2 A - \cot^2 A = 1$$

SIGNS AND VARIATIONS OF TRIGONOMETRIC FUNCTIONS

Quadrant	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
I	+ 0 to 1	+ 1 to 0	+ 0 to ∞	+ ∞ to 0	+ 1 to ∞	+ ∞ to 1
II	+ 1 to 0	− 0 to −1	− $-\infty$ to 0	− 0 to $-\infty$	− $-\infty$ to −1	+ 1 to ∞
III	− 0 to −1	− −1 to 0	+ 0 to ∞	+ ∞ to 0	− −1 to $-\infty$	− $-\infty$ to −1
IV	− −1 to 0	+ 0 to 1	− $-\infty$ to 0	− 0 to $-\infty$	+ ∞ to 1	− −1 to $-\infty$

EXACT VALUES FOR TRIGONOMETRIC FUNCTIONS OF VARIOUS ANGLES

Angle A in degrees	Angle A in radians	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
0°	0	0	1	0	∞	1	∞
15°	$\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
30°	$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
75°	$5\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
90°	$\pi/2$	1	0	$\pm\infty$	0	$\pm\infty$	1
105°	$7\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$-(\sqrt{6}+\sqrt{2})$	$\sqrt{6}-\sqrt{2}$
120°	$2\pi/3$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$3\pi/4$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$5\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
165°	$11\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$-(\sqrt{6}-\sqrt{2})$	$\sqrt{6}+\sqrt{2}$
180°	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
195°	$13\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$-(\sqrt{6}-\sqrt{2})$	$-(\sqrt{6}+\sqrt{2})$
210°	$7\pi/6$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$5\pi/4$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$4\pi/3$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
255°	$17\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$-(\sqrt{6}+\sqrt{2})$	$-(\sqrt{6}-\sqrt{2})$
270°	$3\pi/2$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
285°	$19\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$\sqrt{6}+\sqrt{2}$	$-(\sqrt{6}-\sqrt{2})$
300°	$5\pi/3$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$7\pi/4$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$11\pi/6$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
345°	$23\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$\sqrt{6}-\sqrt{2}$	$-(\sqrt{6}+\sqrt{2})$
360°	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

For tables involving other angles see pages 206-211 and 212-215.

GRAPHS OF TRIGONOMETRIC FUNCTIONS

In each graph x is in radians.

5.22 $y = \sin x$

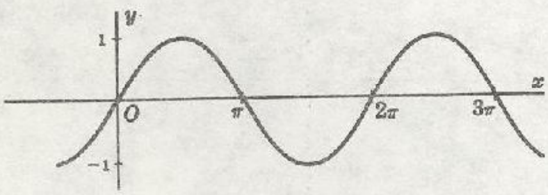

Fig. 5-5

5.23 $y = \cos x$

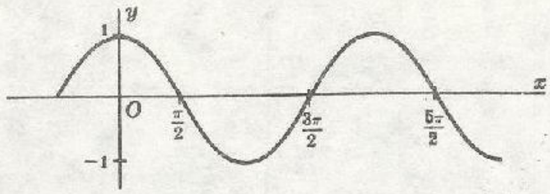

Fig. 5-6

5.24 $y = \tan x$

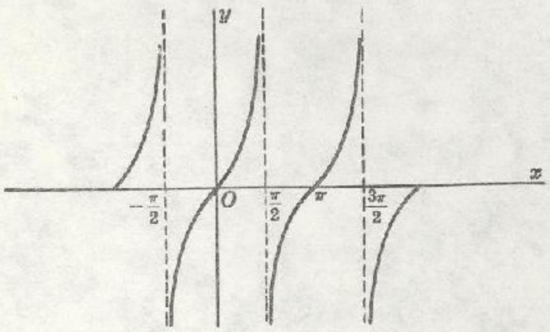

Fig. 5-7

5.25 $y = \cot x$

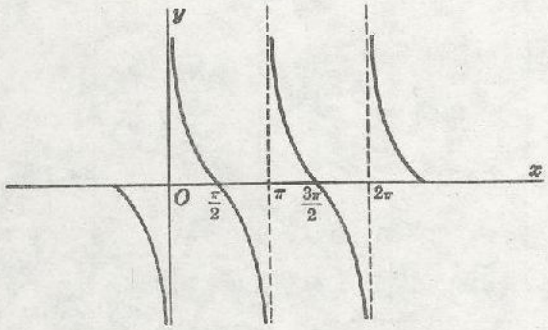

Fig. 5-8

5.26 $y = \sec x$

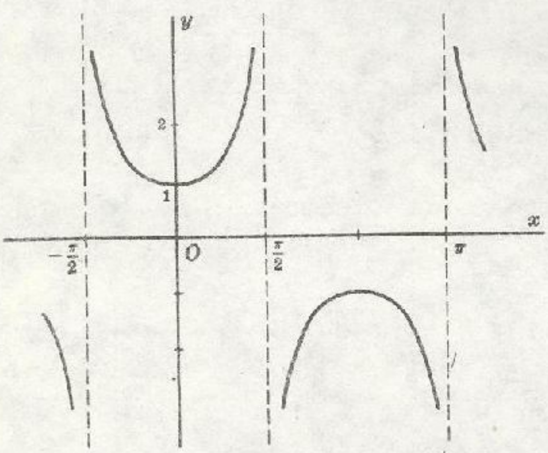

Fig. 5-9

5.27 $y = \csc x$

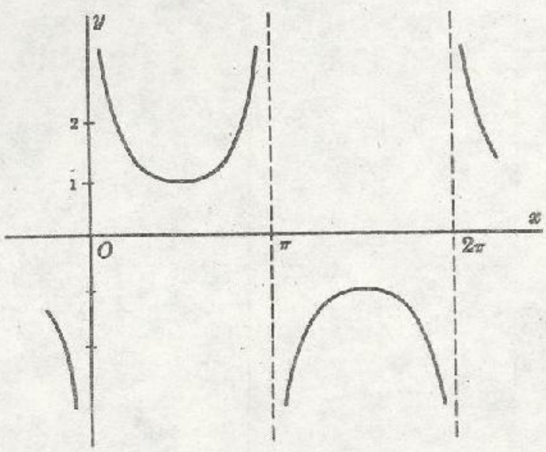

Fig. 5-10

FUNCTIONS OF NEGATIVE ANGLES

5.28 $\sin(-A) = -\sin A$ **5.29** $\cos(-A) = \cos A$ **5.30** $\tan(-A) = -\tan A$

5.31 $\csc(-A) = -\csc A$ **5.32** $\sec(-A) = \sec A$ **5.33** $\cot(-A) = -\cot A$

ADDITION FORMULAS

5.34 $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

5.35 $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

5.36 $$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

5.37 $$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

FUNCTIONS OF ANGLES IN ALL QUADRANTS IN TERMS OF THOSE IN QUADRANT I

	$-A$	$90° \pm A$ $\frac{\pi}{2} \pm A$	$180° \pm A$ $\pi \pm A$	$270° \pm A$ $\frac{3\pi}{2} \pm A$	$k(360°) \pm A$ $2k\pi \pm A$ $k =$ integer
sin	$-\sin A$	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$
cos	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$	$\cos A$
tan	$-\tan A$	$\mp \cot A$	$\pm \tan A$	$\mp \cot A$	$\pm \tan A$
csc	$-\csc A$	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$
sec	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$	$\sec A$
cot	$-\cot A$	$\mp \tan A$	$\pm \cot A$	$\mp \tan A$	$\pm \cot A$

RELATIONSHIPS AMONG FUNCTIONS OF ANGLES IN QUADRANT I

	$\sin A = u$	$\cos A = u$	$\tan A = u$	$\cot A = u$	$\sec A = u$	$\csc A = u$
$\sin A$	u	$\sqrt{1-u^2}$	$u/\sqrt{1+u^2}$	$1/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	$1/u$
$\cos A$	$\sqrt{1-u^2}$	u	$1/\sqrt{1+u^2}$	$u/\sqrt{1+u^2}$	$1/u$	$\sqrt{u^2-1}/u$
$\tan A$	$u/\sqrt{1-u^2}$	$\sqrt{1-u^2}/u$	u	$1/u$	$\sqrt{u^2-1}$	$1/\sqrt{u^2-1}$
$\cot A$	$\sqrt{1-u^2}/u$	$u/\sqrt{1-u^2}$	$1/u$	u	$1/\sqrt{u^2-1}$	$\sqrt{u^2-1}$
$\sec A$	$1/\sqrt{1-u^2}$	$1/u$	$\sqrt{1+u^2}$	$\sqrt{1+u^2}/u$	u	$u/\sqrt{u^2-1}$
$\csc A$	$1/u$	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}/u$	$\sqrt{1+u^2}$	$u/\sqrt{u^2-1}$	u

For extensions to other quadrants use appropriate signs as given in the preceding table.

DOUBLE ANGLE FORMULAS

5.38 $$\sin 2A = 2 \sin A \cos A$$

5.39 $$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

5.40 $$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

HALF ANGLE FORMULAS

5.41 $$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \quad \begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or II} \\ - \text{ if } A/2 \text{ is in quadrant III or IV} \end{bmatrix}$$

5.42 $$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \quad \begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or IV} \\ - \text{ if } A/2 \text{ is in quadrant II or III} \end{bmatrix}$$

5.43 $$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad \begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or III} \\ - \text{ if } A/2 \text{ is in quadrant II or IV} \end{bmatrix}$$

$$= \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} = \csc A - \cot A$$

MULTIPLE ANGLE FORMULAS

5.44 $$\sin 3A = 3\sin A - 4\sin^3 A$$

5.45 $$\cos 3A = 4\cos^3 A - 3\cos A$$

5.46 $$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

5.47 $$\sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$$

5.48 $$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$$

5.49 $$\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

5.50 $$\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A$$

5.51 $$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$$

5.52 $$\tan 5A = \frac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}$$

See also formulas 5.68 and 5.69.

POWERS OF TRIGONOMETRIC FUNCTIONS

5.53 $$\sin^2 A = \tfrac{1}{2} - \tfrac{1}{2}\cos 2A$$

5.54 $$\cos^2 A = \tfrac{1}{2} + \tfrac{1}{2}\cos 2A$$

5.55 $$\sin^3 A = \tfrac{3}{4}\sin A - \tfrac{1}{4}\sin 3A$$

5.56 $$\cos^3 A = \tfrac{3}{4}\cos A + \tfrac{1}{4}\cos 3A$$

5.57 $$\sin^4 A = \tfrac{3}{8} - \tfrac{1}{2}\cos 2A + \tfrac{1}{8}\cos 4A$$

5.58 $$\cos^4 A = \tfrac{3}{8} + \tfrac{1}{2}\cos 2A + \tfrac{1}{8}\cos 4A$$

5.59 $$\sin^5 A = \tfrac{5}{8}\sin A - \tfrac{5}{16}\sin 3A + \tfrac{1}{16}\sin 5A$$

5.60 $$\cos^5 A = \tfrac{5}{8}\cos A + \tfrac{5}{16}\cos 3A + \tfrac{1}{16}\cos 5A$$

See also formulas 5.70 through 5.73.

SUM, DIFFERENCE AND PRODUCT OF TRIGONOMETRIC FUNCTIONS

5.61 $$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)$$

5.62 $$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B)$$

5.63 $$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)$$

5.64 $$\cos A - \cos B = 2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(B-A)$$

5.65 $$\sin A \sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

5.66 $$\cos A \cos B = \tfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

5.67 $$\sin A \cos B = \tfrac{1}{2}\{\sin(A-B) + \sin(A+B)\}$$

GENERAL FORMULAS

5.68 $$\sin nA = \sin A \left\{(2\cos A)^{n-1} - \binom{n-2}{1}(2\cos A)^{n-3} + \binom{n-3}{2}(2\cos A)^{n-5} - \cdots\right\}$$

5.69 $$\cos nA = \frac{1}{2}\left\{(2\cos A)^{n} - \frac{n}{1}(2\cos A)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos A)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos A)^{n-6} + \cdots\right\}$$

5.70 $$\sin^{2n-1} A = \frac{(-1)^{n-1}}{2^{2n-2}}\left\{\sin(2n-1)A - \binom{2n-1}{1}\sin(2n-3)A + \cdots (-1)^{n-1}\binom{2n-1}{n-1}\sin A\right\}$$

5.71 $$\cos^{2n-1} A = \frac{1}{2^{2n-2}}\left\{\cos(2n-1)A + \binom{2n-1}{1}\cos(2n-3)A + \cdots + \binom{2n-1}{n-1}\cos A\right\}$$

5.72 $$\sin^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\left\{\cos 2nA - \binom{2n}{1}\cos(2n-2)A + \cdots (-1)^{n-1}\binom{2n}{n-1}\cos 2A\right\}$$

5.73 $$\cos^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\left\{\cos 2nA + \binom{2n}{1}\cos(2n-2)A + \cdots + \binom{2n}{n-1}\cos 2A\right\}$$

INVERSE TRIGONOMETRIC FUNCTIONS

If $x = \sin y$ then $y = \sin^{-1} x$, i.e. *the angle whose sine is x* or *inverse sine of x*, is a many-valued function of x which is a collection of single-valued functions called *branches*. Similarly the other inverse trigonometric functions are multiple-valued.

For many purposes a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values*.

PRINCIPAL VALUES FOR INVERSE TRIGONOMETRIC FUNCTIONS

Principal values for $x \geqq 0$	Principal values for $x < 0$
$0 \leqq \sin^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \sin^{-1} x < 0$
$0 \leqq \cos^{-1} x \leqq \pi/2$	$\pi/2 < \cos^{-1} x \leqq \pi$
$0 \leqq \tan^{-1} x < \pi/2$	$-\pi/2 < \tan^{-1} x < 0$
$0 < \cot^{-1} x \leqq \pi/2$	$\pi/2 < \cot^{-1} x < \pi$
$0 \leqq \sec^{-1} x < \pi/2$	$\pi/2 < \sec^{-1} x \leqq \pi$
$0 < \csc^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \csc^{-1} x < 0$

RELATIONS BETWEEN INVERSE TRIGONOMETRIC FUNCTIONS

In all cases it is assumed that principal values are used.

5.74 $\sin^{-1} x + \cos^{-1} x = \pi/2$

5.75 $\tan^{-1} x + \cot^{-1} x = \pi/2$

5.76 $\sec^{-1} x + \csc^{-1} x = \pi/2$

5.77 $\csc^{-1} x = \sin^{-1}(1/x)$

5.78 $\sec^{-1} x = \cos^{-1}(1/x)$

5.79 $\cot^{-1} x = \tan^{-1}(1/x)$

5.80 $\sin^{-1}(-x) = -\sin^{-1} x$

5.81 $\cos^{-1}(-x) = \pi - \cos^{-1} x$

5.82 $\tan^{-1}(-x) = -\tan^{-1} x$

5.83 $\cot^{-1}(-x) = \pi - \cot^{-1} x$

5.84 $\sec^{-1}(-x) = \pi - \sec^{-1} x$

5.85 $\csc^{-1}(-x) = -\csc^{-1} x$

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

In each graph y is in radians. Solid portions of curves correspond to principal values.

5.86 $y = \sin^{-1} x$

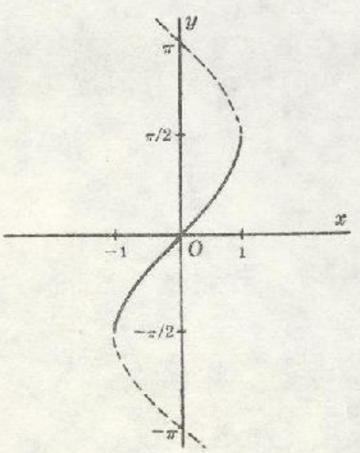

Fig. 5-11

5.87 $y = \cos^{-1} x$

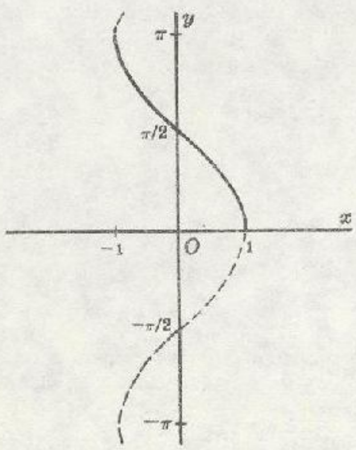

Fig. 5-12

5.88 $y = \tan^{-1} x$

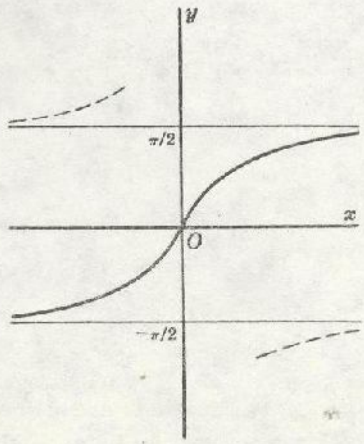

Fig. 5-13

5.89 $y = \cot^{-1} x$

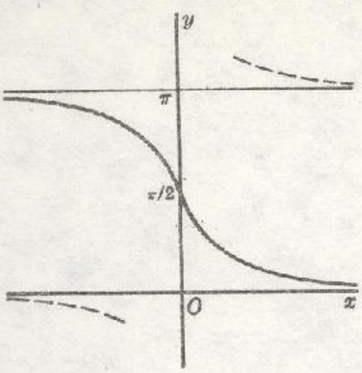

Fig. 5-14

5.90 $y = \sec^{-1} x$

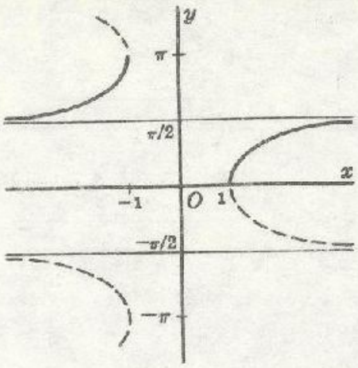

Fig. 5-15

5.91 $y = \csc^{-1} x$

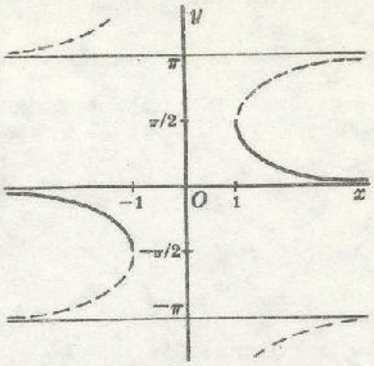

Fig. 5-16

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A PLANE TRIANGLE

The following results hold for any plane triangle ABC with sides a, b, c and angles A, B, C.

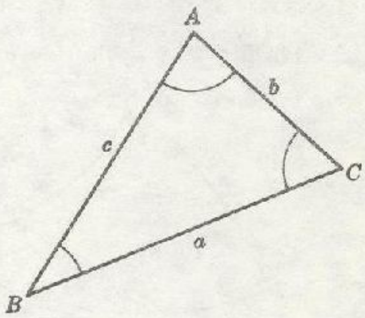

Fig. 5-17

5.92 Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

5.93 Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

with similar relations involving the other sides and angles.

5.94 Law of Tangents

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+P)}{\tan \frac{1}{2}(A-B)}$$

with similar relations involving the other sides and angles.

5.95

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. Similar relations involving angles B and C can be obtained.

See also formulas 4.5, page 5; 4.15 and 4.16, page 6.

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A SPHERICAL TRIANGLE

Spherical triangle ABC is on the surface of a sphere as shown in Fig. 5-18. Sides a, b, c [which are arcs of great circles] are measured by their angles subtended at center O of the sphere. A, B, C are the angles opposite sides a, b, c respectively. Then the following results hold.

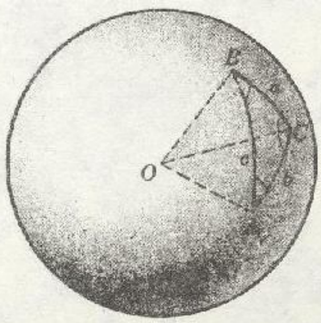

Fig. 5-18

5.96 Law of Sines

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

5.97 Law of Cosines

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

with similar results involving other sides and angles.

5.98 **Law of Tangents**

$$\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

with similar results involving other sides and angles.

5.99

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s-c)}{\sin b \sin c}}$$

where $s = \frac{1}{2}(a+b+c)$. Similar results hold for other sides and angles.

5.100

$$\cos \frac{a}{2} = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}$$

where $S = \frac{1}{2}(A+B+C)$. Similar results hold for other sides and angles.

See also formula 4.44, page 10.

NAPIER'S RULES FOR RIGHT ANGLED SPHERICAL TRIANGLES

Except for right angle C, there are five parts of spherical triangle ABC which if arranged in the order as given in Fig. 5-19 would be a, b, A, c, B.

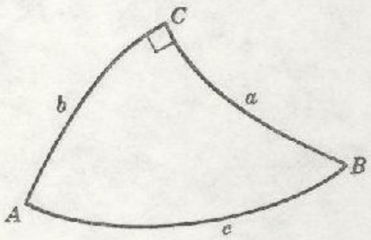

Fig. 5-19

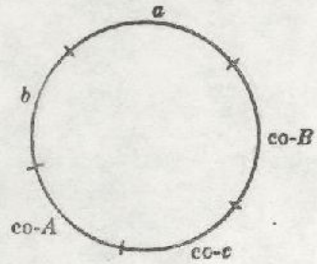

Fig. 5-20

Suppose these quantities are arranged in a circle as in Fig. 5-20 where we attach the prefix co [indicating *complement*] to hypotenuse c and angles A and B.

Any one of the parts of this circle is called a *middle part*, the two neighboring parts are called *adjacent parts* and the two remaining parts are called *opposite parts*. Then Napier's rules are

5.101 The sine of any middle part equals the product of the tangents of the adjacent parts.

5.102 The sine of any middle part equals the product of the cosines of the opposite parts.

Example: Since co-$A = 90° - A$, co-$B = 90° - B$, we have

$$\sin a = \tan b \tan(\text{co-}B) \quad \text{or} \quad \sin a = \tan b \cot B$$

$$\sin(\text{co-}A) = \cos a \cos(\text{co-}B) \quad \text{or} \quad \cos A = \cos a \sin B$$

These can of course be obtained also from the results 5.97 on page 19.

6 COMPLEX NUMBERS

DEFINITIONS INVOLVING COMPLEX NUMBERS

A *complex number* is generally written as $a + bi$ where a and b are real numbers and i, called the *imaginary unit*, has the property that $i^2 = -1$. The real numbers a and b are called the *real* and *imaginary parts* of $a + bi$ respectively.

The complex numbers $a + bi$ and $a - bi$ are called *complex conjugates* of each other.

EQUALITY OF COMPLEX NUMBERS

6.1 $$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

ADDITION OF COMPLEX NUMBERS

6.2 $$(a + bi) + (c + di) = (a + c) + (b + d)i$$

SUBTRACTION OF COMPLEX NUMBERS

6.3 $$(a + bi) - (c + di) = (a - c) + (b - d)i$$

MULTIPLICATION OF COMPLEX NUMBERS

6.4 $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

DIVISION OF COMPLEX NUMBERS

6.5 $$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i$$

Note that the above operations are obtained by using the ordinary rules of algebra and replacing i^2 by -1 wherever it occurs.

GRAPH OF A COMPLEX NUMBER

A complex number $a + bi$ can be plotted as a point (a, b) on an xy plane called an *Argand diagram* or *Gaussian plane*. For example in Fig. 6-1 P represents the complex number $-3 + 4i$.

A complex number can also be interpreted as a *vector* from O to P.

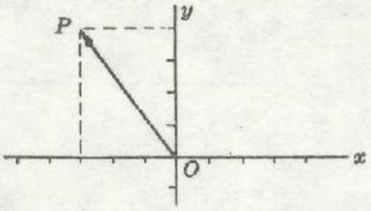

Fig. 6-1

POLAR FORM OF A COMPLEX NUMBER

In Fig. 6-2 point P with coordinates (x, y) represents the complex number $x + iy$. Point P can also be represented by *polar coordinates* (r, θ). Since $x = r\cos\theta$, $y = r\sin\theta$ we have

6.6 $$x + iy = r(\cos\theta + i\sin\theta)$$

called the *polar form* of the complex number. We often call $r = \sqrt{x^2 + y^2}$ the *modulus* and θ the *amplitude* of $x + iy$.

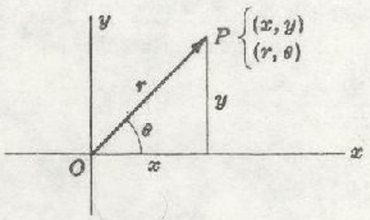

Fig. 6-2

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM

6.7 $$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

6.8 $$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

DE MOIVRE'S THEOREM

If p is any real number, De Moivre's theorem states that

6.9 $$[r(\cos\theta + i\sin\theta)]^p = r^p(\cos p\theta + i\sin p\theta)$$

ROOTS OF COMPLEX NUMBERS

If $p = 1/n$ where n is any positive integer, 6.9 can be written

6.10 $$[r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$$

where k is any integer. From this the n nth roots of a complex number can be obtained by putting $k = 0, 1, 2, \ldots, n-1$.

7 EXPONENTIAL and LOGARITHMIC FUNCTIONS

LAWS OF EXPONENTS

In the following p, q are real numbers, a, b are positive numbers and m, n are positive integers.

7.1	$a^p \cdot a^q = a^{p+q}$	**7.2**	$a^p/a^q = a^{p-q}$	**7.3**	$(a^p)^q = a^{pq}$
7.4	$a^0 = 1,\ a \neq 0$	**7.5**	$a^{-p} = 1/a^p$	**7.6**	$(ab)^p = a^p b^p$
7.7	$\sqrt[n]{a} = a^{1/n}$	**7.8**	$\sqrt[n]{a^m} = a^{m/n}$	**7.9**	$\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$

In a^p, p is called the *exponent*, a is the *base* and a^p is called the pth *power of* a. The function $y = a^x$ is called an *exponential function.*

LOGARITHMS AND ANTILOGARITHMS

If $a^p = N$ where $a \neq 0$ or 1, then $p = \log_a N$ is called the *logarithm* of N to the base a. The number $N = a^p$ is called the *antilogarithm* of p to the base a, written $\text{antilog}_a\, p$.

Example: Since $3^2 = 9$ we have $\log_3 9 = 2$, $\text{antilog}_3 2 = 9$.

The function $y = \log_a x$ is called a *logarithmic function.*

LAWS OF LOGARITHMS

7.10 $$\log_a MN = \log_a M + \log_a N$$

7.11 $$\log_a \frac{M}{N} = \log_a M - \log_a N$$

7.12 $$\log_a M^p = p \log_a M$$

COMMON LOGARITHMS AND ANTILOGARITHMS

Common logarithms and antilogarithms [also called *Briggsian*] are those in which the base $a = 10$. The common logarithm of N is denoted by $\log_{10} N$ or briefly $\log N$. For tables of common logarithms and antilogarithms, see pages 202-205. For illustrations using these tables see pages 194-196.

NATURAL LOGARITHMS AND ANTILOGARITHMS

Natural logarithms and antilogarithms [also called Napierian] are those in which the base $a = e = 2.71828\,18\ldots$ [see page 1]. The natural logarithm of N is denoted by $\log_e N$ or $\ln N$. For tables of natural logarithms see pages 224-225. For tables of natural antilogarithms [i.e. tables giving e^x for values of x] see pages 226-227. For illustrations using these tables see pages 196 and 200.

CHANGE OF BASE OF LOGARITHMS

The relationship between logarithms of a number N to different bases a and b is given by

7.13 $$\log_a N = \frac{\log_b N}{\log_b a}$$

In particular,

7.14 $$\log_e N = \ln N = 2.30258\,50929\,94\ldots \log_{10} N$$

7.15 $$\log_{10} N = \log N = 0.43429\,44819\,03\ldots \log_e N$$

RELATIONSHIP BETWEEN EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

7.16 $$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

These are called *Euler's identities*. Here i is the imaginary unit [see page 21].

7.17 $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

7.18 $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

7.19 $$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

7.20 $$\cot\theta = i\left(\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}\right)$$

7.21 $$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$$

7.22 $$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$$

PERIODICITY OF EXPONENTIAL FUNCTIONS

7.23 $$e^{i(\theta + 2k\pi)} = e^{i\theta} \qquad k = \text{integer}$$

From this it is seen that e^z has period $2\pi i$.

POLAR FORM OF COMPLEX NUMBERS EXPRESSED AS AN EXPONENTIAL

The polar form of a complex number $x + iy$ can be written in terms of exponentials [see 6.6, page 22] as

7.24 $$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

OPERATIONS WITH COMPLEX NUMBERS IN POLAR FORM

Formulas 6.7 through 6.10 on page 22 are equivalent to the following.

7.25 $$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$$

7.26 $$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

7.27 $$(re^{i\theta})^p = r^pe^{ip\theta} \qquad \text{[De Moivre's theorem]}$$

7.28 $$(re^{i\theta})^{1/n} = [re^{i(\theta+2k\pi)}]^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$$

LOGARITHM OF A COMPLEX NUMBER

7.29 $$\ln(re^{i\theta}) = \ln r + i\theta + 2k\pi i \qquad k = \text{integer}$$

8 HYPERBOLIC FUNCTIONS

DEFINITION OF HYPERBOLIC FUNCTIONS

8.1 *Hyperbolic sine of* x $= \sinh x = \dfrac{e^x - e^{-x}}{2}$

8.2 *Hyperbolic cosine of* x $= \cosh x = \dfrac{e^x + e^{-x}}{2}$

8.3 *Hyperbolic tangent of* x $= \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

8.4 *Hyperbolic cotangent of* x $= \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

8.5 *Hyperbolic secant of* x $= \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

8.6 *Hyperbolic cosecant of* x $= \operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$

RELATIONSHIPS AMONG HYPERBOLIC FUNCTIONS

8.7 $\tanh x = \dfrac{\sinh x}{\cosh x}$

8.8 $\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$

8.9 $\operatorname{sech} x = \dfrac{1}{\cosh x}$

8.10 $\operatorname{csch} x = \dfrac{1}{\sinh x}$

8.11 $\cosh^2 x - \sinh^2 x = 1$

8.12 $\operatorname{sech}^2 x + \tanh^2 x = 1$

8.13 $\coth^2 x - \operatorname{csch}^2 x = 1$

FUNCTIONS OF NEGATIVE ARGUMENTS

8.14 $\sinh(-x) = -\sinh x$ **8.15** $\cosh(-x) = \cosh x$ **8.16** $\tanh(-x) = -\tanh x$

8.17 $\operatorname{csch}(-x) = -\operatorname{csch} x$ **8.18** $\operatorname{sech}(-x) = \operatorname{sech} x$ **8.19** $\coth(-x) = -\coth x$

ADDITION FORMULAS

8.20 $$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

8.21 $$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

8.22 $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

8.23 $$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

DOUBLE ANGLE FORMULAS

8.24 $$\sinh 2x = 2 \sinh x \cosh x$$

8.25 $$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

8.26 $$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

HALF ANGLE FORMULAS

8.27 $$\sinh \frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}} \qquad [+ \text{ if } x > 0,\ - \text{ if } x < 0]$$

8.28 $$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

8.29 $$\tanh \frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \qquad [+ \text{ if } x > 0,\ - \text{ if } x < 0]$$
$$= \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}$$

MULTIPLE ANGLE FORMULAS

8.30 $$\sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

8.31 $$\cosh 3x = 4\cosh^3 x - 3\cosh x$$

8.32 $$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

8.33 $$\sinh 4x = 8\sinh^3 x \cosh x + 4 \sinh x \cosh x$$

8.34 $$\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1$$

8.35 $$\tanh 4x = \frac{4\tanh x + 4\tanh^3 x}{1 + 6\tanh^2 x + \tanh^4 x}$$

POWERS OF HYPERBOLIC FUNCTIONS

8.36 $$\sinh^2 x = \tfrac{1}{2}\cosh 2x - \tfrac{1}{2}$$

8.37 $$\cosh^2 x = \tfrac{1}{2}\cosh 2x + \tfrac{1}{2}$$

8.38 $$\sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x$$

8.39 $$\cosh^3 x = \tfrac{1}{4}\cosh 3x + \tfrac{3}{4}\cosh x$$

8.40 $$\sinh^4 x = \tfrac{3}{8} - \tfrac{1}{2}\cosh 2x + \tfrac{1}{8}\cosh 4x$$

8.41 $$\cosh^4 x = \tfrac{3}{8} + \tfrac{1}{2}\cosh 2x + \tfrac{1}{8}\cosh 4x$$

SUM, DIFFERENCE AND PRODUCT OF HYPERBOLIC FUNCTIONS

8.42 $$\sinh x + \sinh y = 2\sinh \tfrac{1}{2}(x+y)\cosh \tfrac{1}{2}(x-y)$$

8.43 $$\sinh x - \sinh y = 2\cosh \tfrac{1}{2}(x+y)\sinh \tfrac{1}{2}(x-y)$$

8.44 $$\cosh x + \cosh y = 2\cosh \tfrac{1}{2}(x+y)\cosh \tfrac{1}{2}(x-y)$$

8.45 $$\cosh x - \cosh y = 2\sinh \tfrac{1}{2}(x+y)\sinh \tfrac{1}{2}(x-y)$$

8.46 $$\sinh x \sinh y = \tfrac{1}{2}\{\cosh(x+y) - \cosh(x-y)\}$$

8.47 $$\cosh x \cosh y = \tfrac{1}{2}\{\cosh(x+y) + \cosh(x-y)\}$$

8.48 $$\sinh x \cosh y = \tfrac{1}{2}\{\sinh(x+y) + \sinh(x-y)\}$$

EXPRESSION OF HYPERBOLIC FUNCTIONS IN TERMS OF OTHERS

In the following we assume $x > 0$. If $x < 0$ use the appropriate sign as indicated by formulas 8.14 to 8.19.

	$\sinh x = u$	$\cosh x = u$	$\tanh x = u$	$\coth x = u$	$\operatorname{sech} x = u$	$\operatorname{csch} x = u$
$\sinh x$	u	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$1/u$
$\cosh x$	$\sqrt{1+u^2}$	u	$1/\sqrt{1-u^2}$	$u/\sqrt{u^2-1}$	$1/u$	$\sqrt{1+u^2}/u$
$\tanh x$	$u/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	u	$1/u$	$\sqrt{1-u^2}$	$1/\sqrt{1+u^2}$
$\coth x$	$\sqrt{u^2+1}/u$	$u/\sqrt{u^2-1}$	$1/u$	u	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}$
$\operatorname{sech} x$	$1/\sqrt{1+u^2}$	$1/u$	$\sqrt{1-u^2}$	$\sqrt{u^2-1}/u$	u	$u/\sqrt{1+u^2}$
$\operatorname{csch} x$	$1/u$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	u

GRAPHS OF HYPERBOLIC FUNCTIONS

8.49 $y = \sinh x$

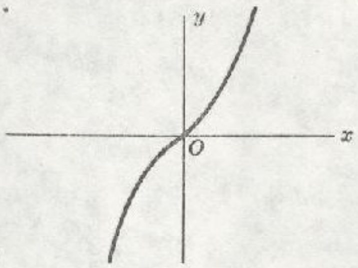

Fig. 8-1

8.50 $y = \cosh x$

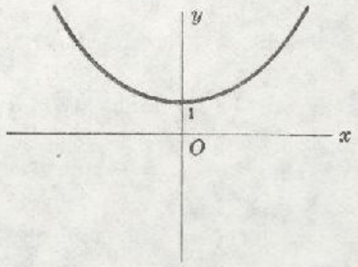

Fig. 8-2

8.51 $y = \tanh x$

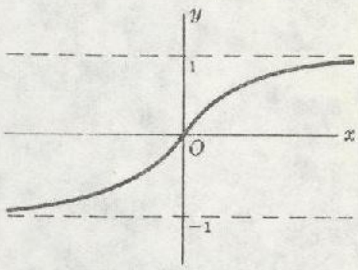

Fig. 8-3

8.52 $y = \coth x$

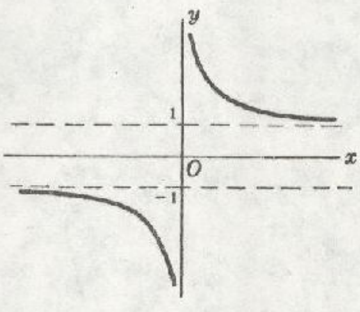

Fig. 8-4

8.53 $y = \operatorname{sech} x$

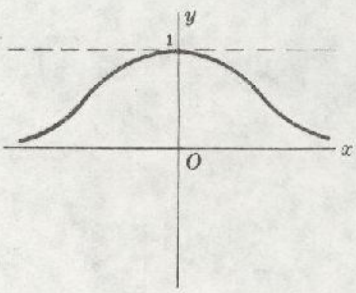

Fig. 8-5

8.54 $y = \operatorname{csch} x$

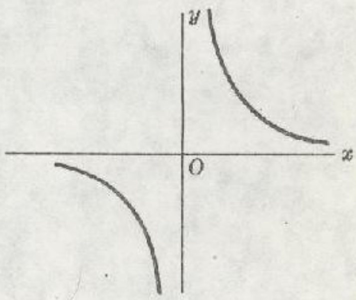

Fig. 8-6

INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh y$, then $y = \sinh^{-1} x$ is called the *inverse hyperbolic sine* of x. Similarly we define the other inverse hyperbolic functions. The inverse hyperbolic functions are multiple-valued and as in the case of inverse trigonometric functions [see page 17] we restrict ourselves to principal values for which they can be considered as single-valued.

The following list shows the principal values [unless otherwise indicated] of the inverse hyperbolic functions expressed in terms of logarithmic functions which are taken as real valued.

8.55 $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad -\infty < x < \infty$

8.56 $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad x \geqq 1 \qquad [\cosh^{-1} x > 0$ is principal value$]$

8.57 $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$

8.58 $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad x > 1$ or $x < -1$

8.59 $\operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) \qquad 0 < x \leqq 1 \qquad [\operatorname{sech}^{-1} x > 0$ is principal value$]$

8.60 $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) \qquad x \neq 0$

RELATIONS BETWEEN INVERSE HYPERBOLIC FUNCTIONS

8.61 $$\operatorname{csch}^{-1} x = \sinh^{-1}(1/x)$$

8.62 $$\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$$

8.63 $$\coth^{-1} x = \tanh^{-1}(1/x)$$

8.64 $$\sinh^{-1}(-x) = -\sinh^{-1} x$$

8.65 $$\tanh^{-1}(-x) = -\tanh^{-1} x$$

8.66 $$\coth^{-1}(-x) = -\coth^{-1} x$$

8.67 $$\operatorname{csch}^{-1}(-x) = -\operatorname{csch}^{-1} x$$

GRAPHS OF INVERSE HYPERBOLIC FUNCTIONS

8.68 $y = \sinh^{-1} x$

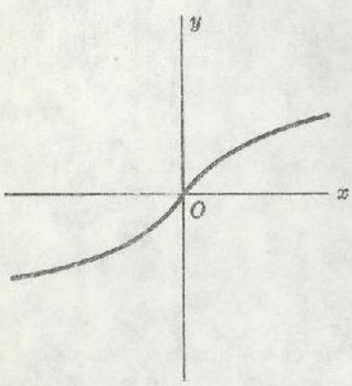

Fig. 8-7

8.69 $y = \cosh^{-1} x$

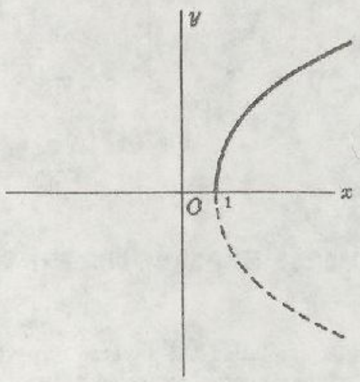

Fig. 8-8

8.70 $y = \tanh^{-1} x$

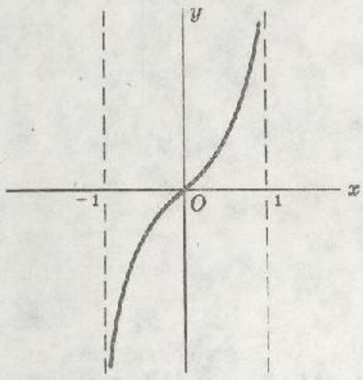

Fig. 8-9

8.71 $y = \coth^{-1} x$

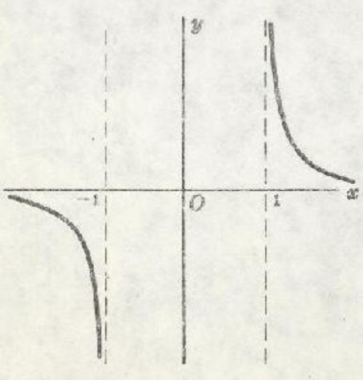

Fig. 8-10

8.72 $y = \operatorname{sech}^{-1} x$

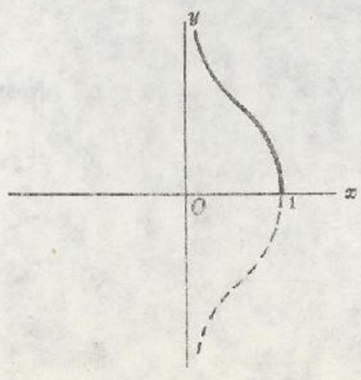

Fig. 8-11

8.73 $y = \operatorname{csch}^{-1} x$

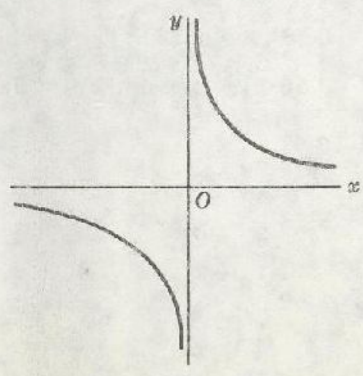

Fig. 8-12

RELATIONSHIP BETWEEN HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS

8.74 $\sin(ix) = i\sinh x$

8.75 $\cos(ix) = \cosh x$

8.76 $\tan(ix) = i\tanh x$

8.77 $\csc(ix) = -i\,\text{csch}\, x$

8.78 $\sec(ix) = \text{sech}\, x$

8.79 $\cot(ix) = -i\coth x$

8.80 $\sinh(ix) = i\sin x$

8.81 $\cosh(ix) = \cos x$

8.82 $\tanh(ix) = i\tan x$

8.83 $\text{csch}(ix) = -i\csc x$

8.84 $\text{sech}(ix) = \sec x$

8.85 $\coth(ix) = -i\cot x$

PERIODICITY OF HYPERBOLIC FUNCTIONS

In the following k is any integer.

8.86 $\sinh(x + 2k\pi i) = \sinh x$

8.87 $\cosh(x + 2k\pi i) = \cosh x$

8.88 $\tanh(x + k\pi i) = \tanh x$

8.89 $\text{csch}(x + 2k\pi i) = \text{csch}\, x$

8.90 $\text{sech}(x + 2k\pi i) = \text{sech}\, x$

8.91 $\coth(x + k\pi i) = \coth x$

RELATIONSHIP BETWEEN INVERSE HYPERBOLIC AND INVERSE TRIGONOMETRIC FUNCTIONS

8.92 $\sin^{-1}(ix) = i\sinh^{-1} x$

8.93 $\sinh^{-1}(ix) = i\sin^{-1} x$

8.94 $\cos^{-1} x = \pm i\cosh^{-1} x$

8.95 $\cosh^{-1} x = \pm i\cos^{-1} x$

8.96 $\tan^{-1}(ix) = i\tanh^{-1} x$

8.97 $\tanh^{-1}(ix) = i\tan^{-1} x$

8.98 $\cot^{-1}(ix) = -i\coth^{-1} x$

8.99 $\coth^{-1}(ix) = -i\cot^{-1} x$

8.100 $\sec^{-1} x = \pm i\,\text{sech}^{-1} x$

8.101 $\text{sech}^{-1} x = \pm i\sec^{-1} x$

8.102 $\csc^{-1}(ix) = -i\,\text{csch}^{-1} x$

8.103 $\text{csch}^{-1}(ix) = -i\csc^{-1} x$

9 SOLUTIONS of ALGEBRAIC EQUATIONS

QUADRATIC EQUATION: $ax^2 + bx + c = 0$

9.1 Solutions: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If a, b, c are real and if $D = b^2 - 4ac$ is the *discriminant*, then the roots are

(i) real and unequal if $D > 0$

(ii) real and equal if $D = 0$

(iii) complex conjugate if $D < 0$

9.2 If x_1, x_2 are the roots, then $x_1 + x_2 = -b/a$ and $x_1 x_2 = c/a$.

CUBIC EQUATION: $x^3 + a_1x^2 + a_2x + a_3 = 0$

Let $$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54},$$
$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

9.3 Solutions: $$\begin{cases} x_1 = S + T - \frac{1}{3}a_1 \\ x_2 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 + \frac{1}{2}i\sqrt{3}\,(S-T) \\ x_3 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 - \frac{1}{2}i\sqrt{3}\,(S-T) \end{cases}$$

If a_1, a_2, a_3 are real and if $D = Q^3 + R^2$ is the *discriminant*, then

(i) one root is real and two complex conjugate if $D > 0$

(ii) all roots are real and at least two are equal if $D = 0$

(iii) all roots are real and unequal if $D < 0$.

If $D < 0$, computation is simplified by use of trigonometry.

9.4 Solutions if $D < 0$: $$\begin{cases} x_1 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta) - \frac{1}{3}a_1 \\ x_2 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta + 120°) - \frac{1}{3}a_1 \\ x_3 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta + 240°) - \frac{1}{3}a_1 \end{cases} \quad \text{where } \cos\theta = R/\sqrt{-Q^3}$$

9.5 $$x_1 + x_2 + x_3 = -a_1, \quad x_1x_2 + x_2x_3 + x_3x_1 = a_2, \quad x_1x_2x_3 = -a_3$$

where x_1, x_2, x_3 are the three roots.

QUARTIC EQUATION: $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$

Let y_1 be a real root of the cubic equation

9.6 $$y^3 - a_2y^2 + (a_1a_3 - 4a_4)y + (4a_2a_4 - a_3^2 - a_1^2a_4) = 0$$

9.7 Solutions: The 4 roots of $z^2 + \frac{1}{2}\{a_1 \pm \sqrt{a_1^2 - 4a_2 + 4y_1}\}z + \frac{1}{2}\{y_1 \mp \sqrt{y_1^2 - 4a_4}\} = 0$

If all roots of 9.6 are real, computation is simplified by using that particular real root which produces all real coefficients in the quadratic equation 9.7.

9.8 $$\begin{cases} x_1 + x_2 + x_3 + x_4 = -a_1 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 + x_1x_3 + x_2x_4 = a_2 \\ x_1x_2x_3 + x_2x_3x_4 + x_1x_2x_4 + x_1x_3x_4 = -a_3 \\ x_1x_2x_3x_4 = a_4 \end{cases}$$

where x_1, x_2, x_3, x_4 are the four roots.

10 FORMULAS from PLANE ANALYTIC GEOMETRY

DISTANCE d BETWEEN TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

10.1 $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

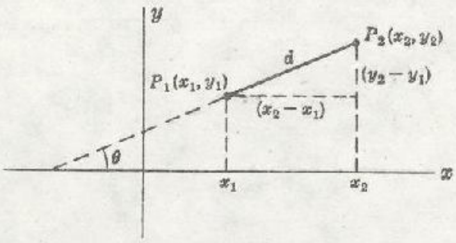

Fig. 10-1

SLOPE m OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

10.2 $$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta$$

EQUATION OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

10.3 $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m \quad \text{or} \quad y - y_1 = m(x - x_1)$$

10.4 $$y = mx + b$$

where $b = y_1 - mx_1 = \dfrac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$ is the *intercept* on the y axis, i.e. the y *intercept*.

EQUATION OF LINE IN TERMS OF x INTERCEPT $a \neq 0$ AND y INTERCEPT $b \neq 0$

10.5 $$\frac{x}{a} + \frac{y}{b} = 1$$

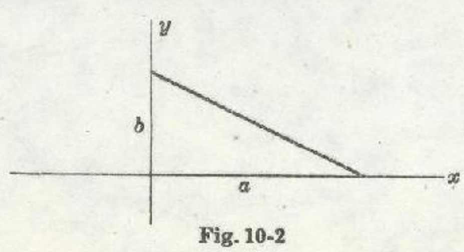

Fig. 10-2

NORMAL FORM FOR EQUATION OF LINE

10.6 $$x\cos\alpha + y\sin\alpha = p$$

where p = perpendicular distance from origin O to line

and α = angle of inclination of perpendicular with positive x axis.

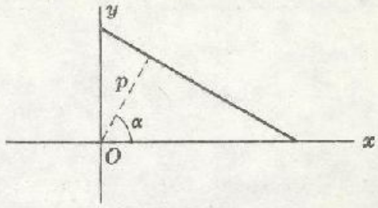

Fig. 10-3

GENERAL EQUATION OF LINE

10.7 $$Ax + By + C = 0$$

DISTANCE FROM POINT (x_1, y_1) TO LINE $Ax + By + C = 0$

10.8 $$\frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

where the sign is chosen so that the distance is nonnegative.

ANGLE ψ BETWEEN TWO LINES HAVING SLOPES m_1 AND m_2

10.9 $$\tan\psi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Lines are parallel or coincident if and only if $m_1 = m_2$.

Lines are perpendicular if and only if $m_2 = -1/m_1$.

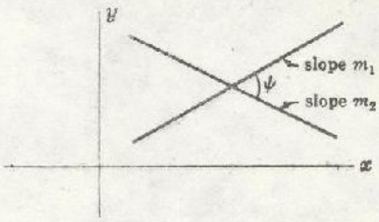

Fig. 10-4

AREA OF TRIANGLE WITH VERTICES AT $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

10.10 $$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}(x_1y_2 + y_1x_3 + y_3x_2 - y_2x_3 - y_1x_2 - x_1y_3)$$

where the sign is chosen so that the area is nonnegative.

If the area is zero the points all lie on a line.

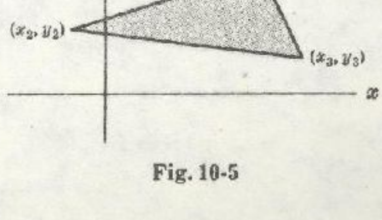

Fig. 10-5

TRANSFORMATION OF COORDINATES INVOLVING PURE TRANSLATION

10.11 $$\begin{cases} x = x' + x_0 \\ y = y' + y_0 \end{cases} \quad \text{or} \quad \begin{cases} x' = x - x_0 \\ y' = y - y_0 \end{cases}$$

where (x, y) are old coordinates [i.e. coordinates relative to xy system], (x', y') are new coordinates [relative to $x'y'$ system] and (x_0, y_0) are the coordinates of the new origin O' relative to the old xy coordinate system.

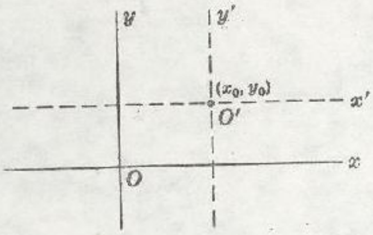

Fig. 10-6

TRANSFORMATION OF COORDINATES INVOLVING PURE ROTATION

10.12 $$\begin{cases} x = x' \cos\alpha - y' \sin\alpha \\ y = x' \sin\alpha + y' \cos\alpha \end{cases} \quad \text{or} \quad \begin{cases} x' = x \cos\alpha + y \sin\alpha \\ y' = y \cos\alpha - x \sin\alpha \end{cases}$$

where the origins of the old $[xy]$ and new $[x'y']$ coordinate systems are the same but the x' axis makes an angle α with the positive x axis.

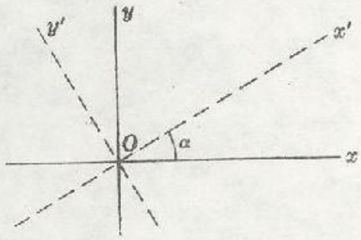

Fig. 10-7

TRANSFORMATION OF COORDINATES INVOLVING TRANSLATION AND ROTATION

10.13 $$\begin{cases} x = x' \cos\alpha - y' \sin\alpha + x_0 \\ y = x' \sin\alpha + y' \cos\alpha + y_0 \end{cases}$$

or $$\begin{cases} x' = (x - x_0) \cos\alpha + (y - y_0) \sin\alpha \\ y' = (y - y_0) \cos\alpha - (x - x_0) \sin\alpha \end{cases}$$

where the new origin O' of $x'y'$ coordinate system has coordinates (x_0, y_0) relative to the old xy coordinate system and the x' axis makes an angle α with the positive x axis.

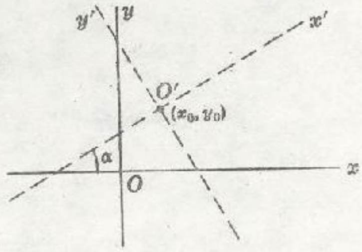

Fig. 10-8

POLAR COORDINATES (r, θ)

A point P can be located by rectangular coordinates (x, y) or polar coordinates (r, θ). The transformation between these coordinates is

10.14 $$\begin{cases} x = r \cos\theta \\ y = r \sin\theta \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \end{cases}$$

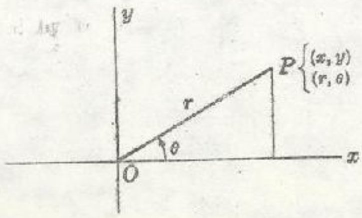

Fig. 10-9

EQUATION OF CIRCLE OF RADIUS R, CENTER AT (x_0, y_0)

10.15 $$(x - x_0)^2 + (y - y_0)^2 = R^2$$

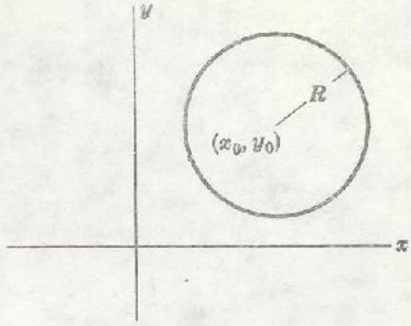

Fig. 10-10

EQUATION OF CIRCLE OF RADIUS R PASSING THROUGH ORIGIN

10.16 $$r = 2R\cos(\theta - \alpha)$$

where (r, θ) are polar coordinates of any point on the circle and (R, α) are polar coordinates of the center of the circle.

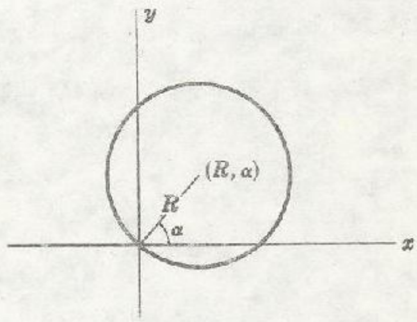

Fig. 10-11

CONICS [ELLIPSE, PARABOLA OR HYPERBOLA]

If a point P moves so that its distance from a fixed point [called the *focus*] divided by its distance from a fixed line [called the *directrix*] is a constant ϵ [called the *eccentricity*], then the curve described by P is called a *conic* [so-called because such curves can be obtained by intersecting a plane and a cone at different angles].

If the focus is chosen at origin O the equation of a conic in polar coordinates (r, θ) is, if $OQ = p$ and $LM = D$, [see Fig. 10-12]

10.17 $$r = \frac{p}{1 - \epsilon\cos\theta} = \frac{\epsilon D}{1 - \epsilon\cos\theta}$$

The conic is

(i) an ellipse if $\epsilon < 1$

(ii) a parabola if $\epsilon = 1$

(iii) a hyperbola if $\epsilon > 1$.

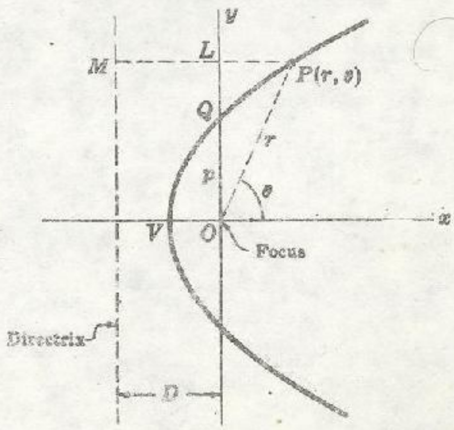

Fig. 10-12

ELLIPSE WITH CENTER $C(x_0, y_0)$ AND MAJOR AXIS PARALLEL TO x AXIS

10.18 Length of major axis $A'A = 2a$

10.19 Length of minor axis $B'B = 2b$

10.20 Distance from center C to focus F or F' is

$$c = \sqrt{a^2 - b^2}$$

10.21 Eccentricity $= \epsilon = \dfrac{c}{a} = \dfrac{\sqrt{a^2-b^2}}{a}$

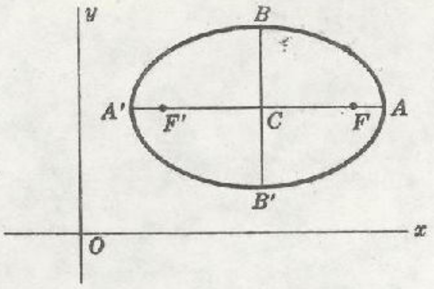

Fig. 10-13

10.22 Equation in rectangular coordinates:

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

10.23 Equation in polar coordinates if C is at O: $\quad r^2 = \dfrac{a^2b^2}{a^2 \sin^2\theta + b^2\cos^2\theta}$

10.24 Equation in polar coordinates if C is on x axis and F' is at O: $\quad r = \dfrac{a(1-\epsilon^2)}{1-\epsilon\cos\theta}$

10.25 If P is any point on the ellipse, $PF + PF' = 2a$

If the major axis is parallel to the y axis, interchange x and y in the above or replace θ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

PARABOLA WITH AXIS PARALLEL TO x AXIS

If vertex is at $A(x_0, y_0)$ and the distance from A to focus F is $a > 0$, the equation of the parabola is

10.26 $\quad (y-y_0)^2 = 4a(x-x_0) \quad$ if parabola opens to right [Fig. 10-14]

10.27 $\quad (y-y_0)^2 = -4a(x-x_0) \quad$ if parabola opens to left [Fig. 10-15]

If focus is at the origin [Fig. 10-16] the equation in polar coordinates is

10.28 $$r = \frac{2a}{1-\cos\theta}$$

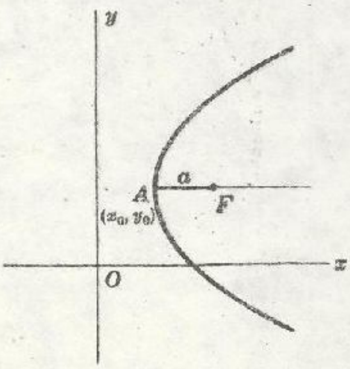

Fig. 10-14

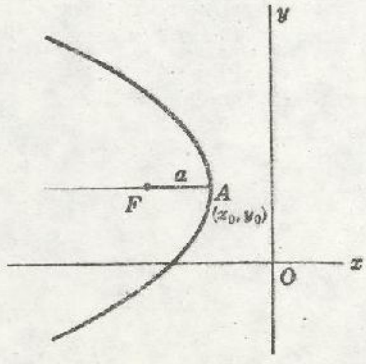

Fig. 10-15

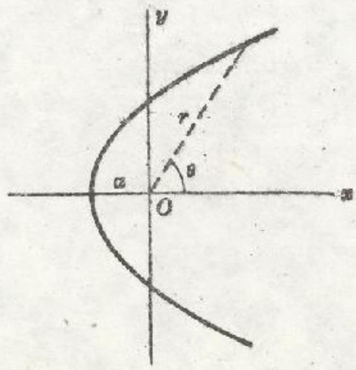

Fig. 10-16

In case the axis is parallel to the y axis, interchange x and y or replace θ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

HYPERBOLA WITH CENTER $C(x_0, y_0)$ AND MAJOR AXIS PARALLEL TO x AXIS

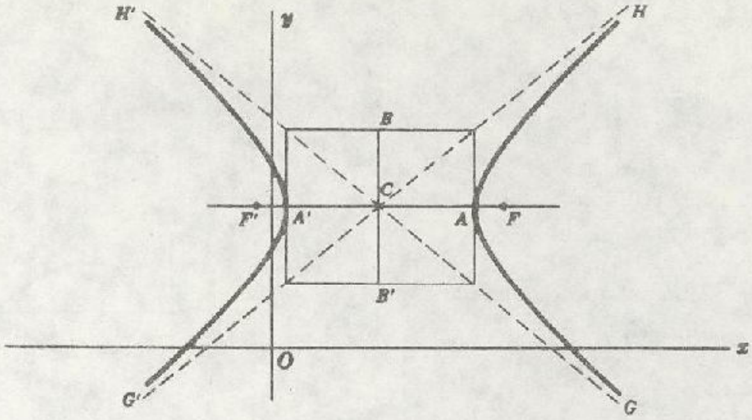

Fig. 10-17

10.29 Length of major axis $A'A = 2a$

10.30 Length of minor axis $B'B = 2b$

10.31 Distance from center C to focus F or $F' = c = \sqrt{a^2 + b^2}$

10.32 Eccentricity $\epsilon = \dfrac{c}{a} = \dfrac{\sqrt{a^2+b^2}}{a}$

10.33 Equation in rectangular coordinates: $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} = 1$

10.34 Slopes of asymptotes $G'H$ and $GH' = \pm \dfrac{b}{a}$

10.35 Equation in polar coordinates if C is at O: $r^2 = \dfrac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta}$

10.36 Equation in polar coordinates if C is on X axis and F' is at O: $r = \dfrac{a(\epsilon^2 - 1)}{1 - \epsilon\cos\theta}$

10.37 If P is any point on the hyperbola, $PF - PF' = \pm 2a$ [depending on branch]

If the major axis is parallel to the y axis, interchange x and y in the above or replace θ by $\frac{1}{2}\pi - \theta$ [or $90^\circ - \theta$].

11 SPECIAL PLANE CURVES

LEMNISCATE

11.1 Equation in polar coordinates:

$$r^2 = a^2 \cos 2\theta$$

11.2 Equation in rectangular coordinates:

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

11.3 Angle between AB' or $A'B$ and x axis $= 45°$

11.4 Area of one loop $= a^2$

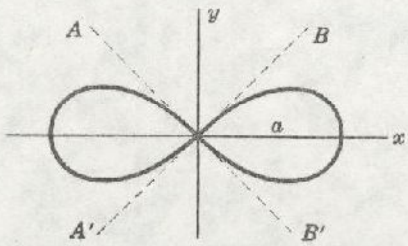

Fig. 11-1

CYCLOID

11.5 Equations in parametric form:

$$\begin{cases} x = a(\phi - \sin \phi) \\ y = a(1 - \cos \phi) \end{cases}$$

11.6 Area of one arch $= 3\pi a^2$

11.7 Arc length of one arch $= 8a$

This is a curve described by a point P on a circle of radius a rolling along x axis.

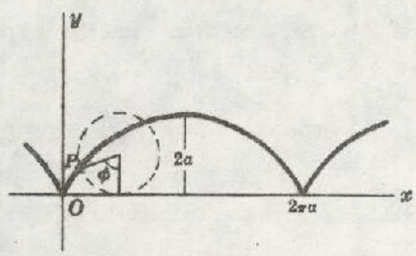

Fig. 11-2

HYPOCYCLOID WITH FOUR CUSPS

11.8 Equation in rectangular coordinates:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

11.9 Equations in parametric form:

$$\begin{cases} x = a \cos^3 \theta \\ y = a \sin^3 \theta \end{cases}$$

11.10 Area bounded by curve $= \frac{3}{8}\pi a^2$

11.11 Arc length of entire curve $= 6a$

This is a curve described by a point P on a circle of radius $a/4$ as it rolls on the inside of a circle of radius a.

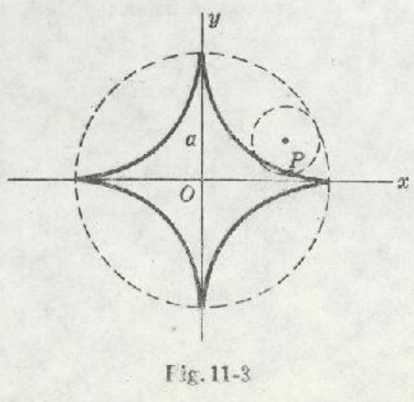

Fig. 11-3

CARDIOID

11.12 Equation: $r = 2a(1 + \cos\theta)$

11.13 Area bounded by curve $= \frac{3}{2}\pi a^2$

11.14 Arc length of curve $= 8a$

This is the curve described by a point P of a circle of radius a as it rolls on the outside of a fixed circle of radius a. The curve is also a special case of the limacon of Pascal [see 11.32].

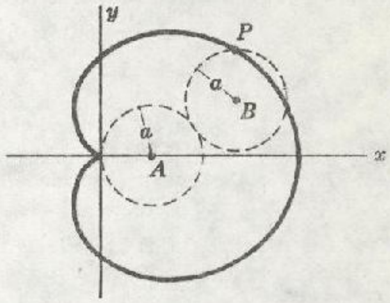

Fig. 11-4

CATENARY

11.15 Equation: $y = \frac{a}{2}(e^{x/a} + e^{-x/a}) = a\cosh\frac{x}{a}$

This is the curve in which a heavy uniform chain would hang if suspended vertically from fixed points A and B.

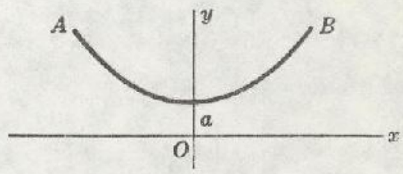

Fig. 11-5

THREE-LEAVED ROSE

11.16 Equation: $r = a\cos 3\theta$

The equation $r = a\sin 3\theta$ is a similar curve obtained by rotating the curve of Fig. 11-6 counterclockwise through 30° or $\pi/6$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has n leaves if n is odd.

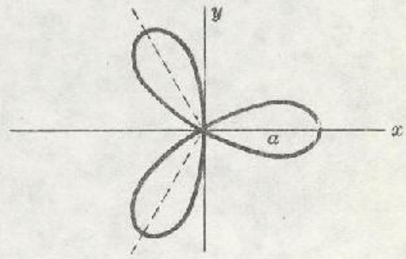

Fig. 11-6

FOUR-LEAVED ROSE

11.17 Equation: $r = a\cos 2\theta$

The equation $r = a\sin 2\theta$ is a similar curve obtained by rotating the curve of Fig. 11-7 counterclockwise through 45° or $\pi/4$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has $2n$ leaves if n is even.

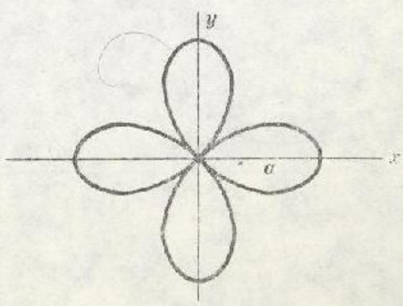

Fig. 11-7