CARDIOID

11.12 Equation: $r = 2a(1 + \cos\theta)$

11.13 Area bounded by curve $= \frac{3}{2}\pi a^2$

11.14 Arc length of curve $= 8a$

This is the curve described by a point P of a circle of radius a as it rolls on the outside of a fixed circle of radius a. The curve is also a special case of the limacon of Pascal [see 11.32].

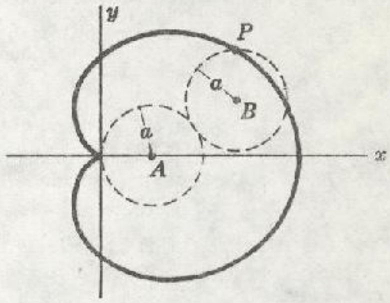

Fig. 11-4

CATENARY

11.15 Equation: $y = \frac{a}{2}(e^{x/a} + e^{-x/a}) = a\cosh\frac{x}{a}$

This is the curve in which a heavy uniform chain would hang if suspended vertically from fixed points A and B.

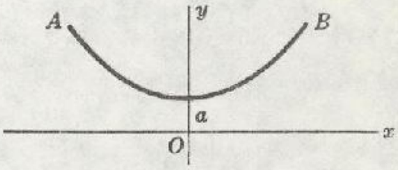

Fig. 11-5

THREE-LEAVED ROSE

11.16 Equation: $r = a\cos 3\theta$

The equation $r = a\sin 3\theta$ is a similar curve obtained by rotating the curve of Fig. 11-6 counterclockwise through 30° or $\pi/6$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has n leaves if n is odd.

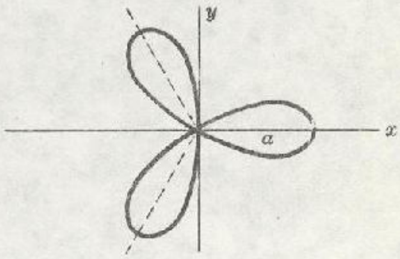

Fig. 11-6

FOUR-LEAVED ROSE

11.17 Equation: $r = a\cos 2\theta$

The equation $r = a\sin 2\theta$ is a similar curve obtained by rotating the curve of Fig. 11-7 counterclockwise through 45° or $\pi/4$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has $2n$ leaves if n is even.

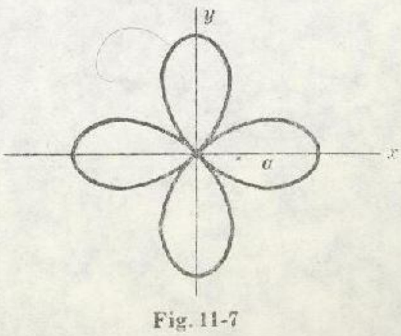

Fig. 11-7

EPICYCLOID

11.18 **Parametric equations:**

$$\begin{cases} x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\right)\theta \\ y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\right)\theta \end{cases}$$

This is the curve described by a point P on a circle of radius b as it rolls on the outside of a circle of radius a.

The cardioid [Fig. 11-4] is a special case of an epicycloid.

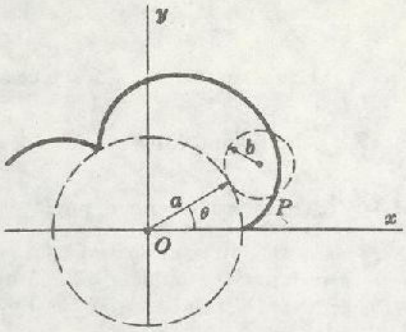

Fig. 11-8

GENERAL HYPOCYCLOID

11.19 **Parametric equations:**

$$\begin{cases} x = (a-b)\cos\phi + b\cos\left(\dfrac{a-b}{b}\right)\phi \\ y = (a-b)\sin\phi - b\sin\left(\dfrac{a-b}{b}\right)\phi \end{cases}$$

This is the curve described by a point P on a circle of radius b as it rolls on the inside of a circle of radius a.

If $b = a/4$, the curve is that of Fig. 11-3.

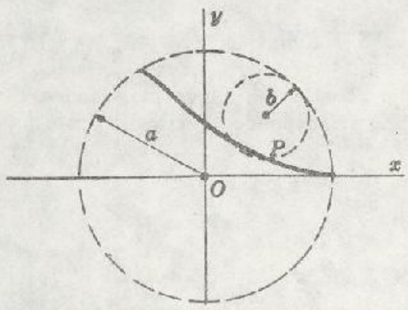

Fig. 11-9

TROCHOID

11.20 Parametric equations: $\begin{cases} x = a\phi - b\sin\phi \\ y = a - b\cos\phi \end{cases}$

This is the curve described by a point P at distance b from the center of a circle of radius a as the circle rolls on the x axis.

If $b < a$, the curve is as shown in Fig. 11-10 and is called a *curtate cycloid*.

If $b > a$, the curve is as shown in Fig. 11-11 and is called a *prolate cycloid*.

If $b = a$, the curve is the cycloid of Fig. 11-2.

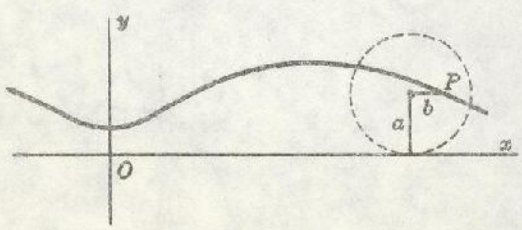

Fig. 11-10

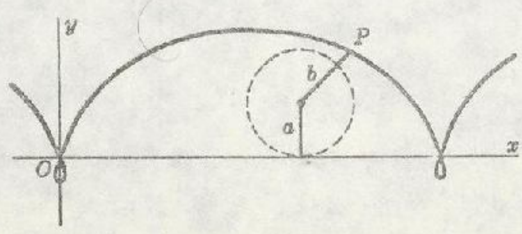

Fig. 11-11

TRACTRIX

11.21 Parametric equations: $\begin{cases} x = a(\ln \cot \frac{1}{2}\phi - \cos \phi) \\ y = a \sin \phi \end{cases}$

This is the curve described by endpoint P of a taut string PQ of length a as the other end Q is moved along the x axis.

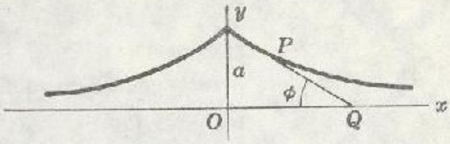

Fig. 11-12

WITCH OF AGNESI

11.22 Equation in rectangular coordinates: $y = \dfrac{8a^3}{x^2 + 4a^2}$

11.23 Parametric equations: $\begin{cases} x = 2a \cot \theta \\ y = a(1 - \cos 2\theta) \end{cases}$

In Fig. 11-13 the variable line OA intersects $y = 2a$ and the circle of radius a with center $(0, a)$ at A and B respectively. Any point P on the "witch" is located by constructing lines parallel to the x and y axes through B and A respectively and determining the point P of intersection.

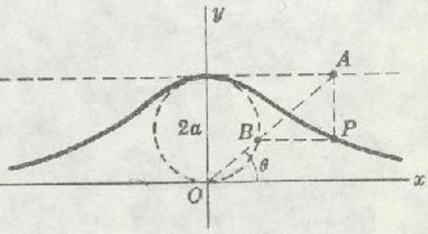

Fig. 11-13

FOLIUM OF DESCARTES

11.24 Equation in rectangular coordinates:

$$x^3 + y^3 = 3axy$$

11.25 Parametric equations:

$$\begin{cases} x = \dfrac{3at}{1 + t^3} \\ y = \dfrac{3at^2}{1 + t^3} \end{cases}$$

11.26 Area of loop $= \frac{3}{2}a^2$

11.27 Equation of asymptote: $x + y + a = 0$

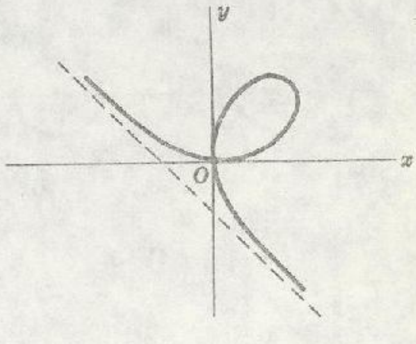

Fig. 11-14

INVOLUTE OF A CIRCLE

11.28 Parametric equations:

$$\begin{cases} x = a(\cos \phi + \phi \sin \phi) \\ y = a(\sin \phi - \phi \cos \phi) \end{cases}$$

This is the curve described by the endpoint P of a string as it unwinds from a circle of radius a while held taut.

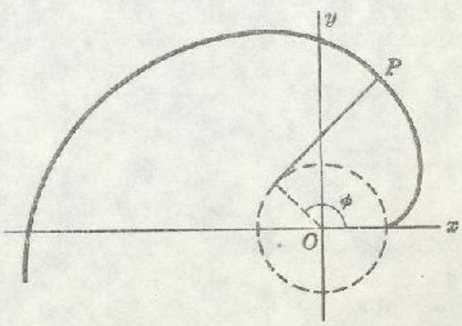

Fig. 11-15

EVOLUTE OF AN ELLIPSE

11.29 Equation in rectangular coordinates:

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

11.30 Parametric equations:

$$\begin{cases} ax = (a^2 - b^2)\cos^3\theta \\ by = (a^2 - b^2)\sin^3\theta \end{cases}$$

This curve is the envelope of the normals to the ellipse $x^2/a^2 + y^2/b^2 = 1$ shown dashed in Fig. 11-16.

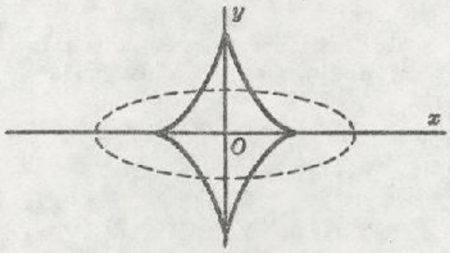

Fig. 11-16

OVALS OF CASSINI

11.31 Polar equation: $r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4$

This is the curve described by a point P such that the product of its distances from two fixed points [distance $2a$ apart] is a constant b^2.

The curve is as in Fig. 11-17 or Fig. 11-18 according as $b < a$ or $b > a$ respectively.

If $b = a$, the curve is a *lemniscate* [Fig. 11-1].

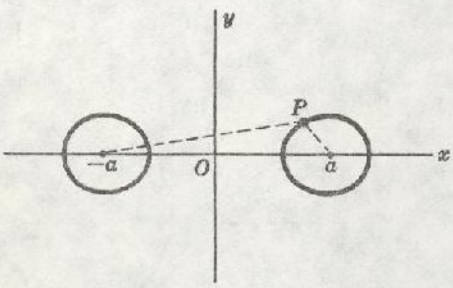

Fig. 11-17

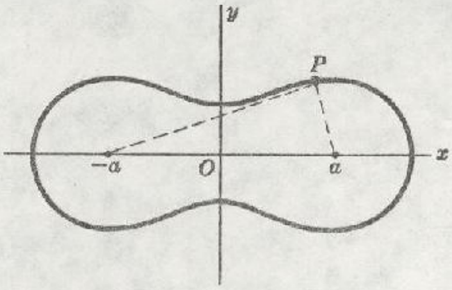

Fig. 11-18

LIMACON OF PASCAL

11.32 Polar equation: $r = b + a\cos\theta$

Let OQ be a line joining origin O to any point Q on a circle of diameter a passing through O. Then the curve is the locus of all points P such that $PQ = b$.

The curve is as in Fig. 11-19 or Fig. 11-20 according as $b > a$ or $b < a$ respectively. If $b = a$, the curve is a *cardioid* [Fig. 11-4].

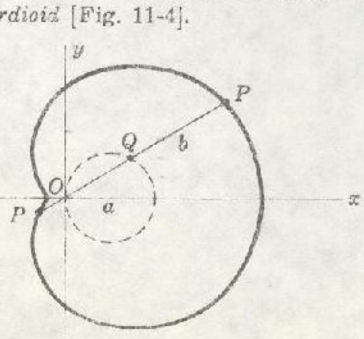

Fig. 11-19

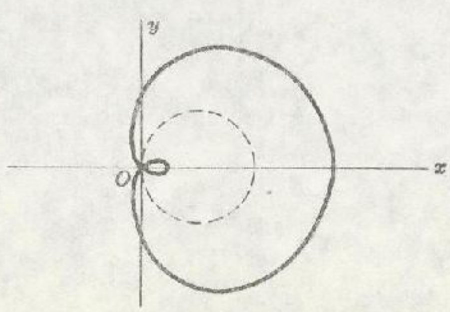

Fig. 11-20

CISSOID OF DIOCLES

11.33 Equation in rectangular coordinates:

$$y^2 = \frac{x^3}{2a - x}$$

11.34 Parametric equations:

$$\begin{cases} x = 2a \sin^2 \theta \\ y = \dfrac{2a \sin^3 \theta}{\cos \theta} \end{cases}$$

This is the curve described by a point P such that the distance OP = distance RS. It is used in the problem of *duplication of a cube*, i.e. finding the side of a cube which has twice the volume of a given cube.

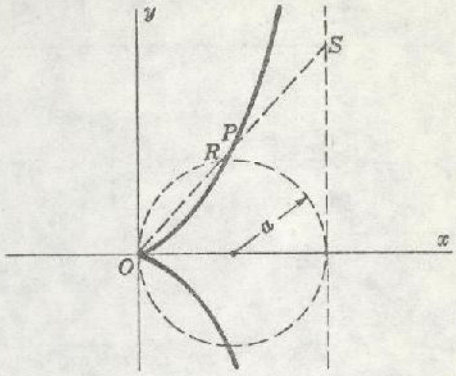

Fig. 11-21

SPIRAL OF ARCHIMEDES

11.35 Polar equation: $r = a\theta$

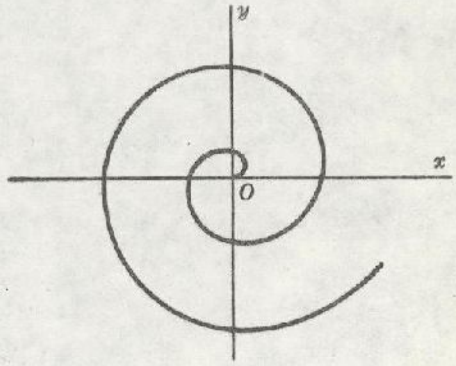

Fig. 11-22

12 FORMULAS from SOLID ANALYTIC GEOMETRY

DISTANCE d BETWEEN TWO POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

12.1 $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

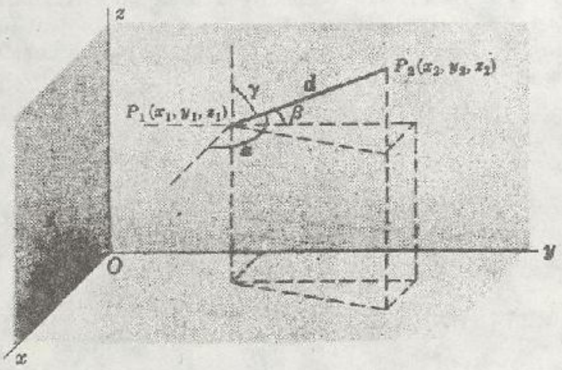

Fig. 12-1

DIRECTION COSINES OF LINE JOINING POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

12.2 $$l = \cos\alpha = \frac{x_2 - x_1}{d}, \quad m = \cos\beta = \frac{y_2 - y_1}{d}, \quad n = \cos\gamma = \frac{z_2 - z_1}{d}$$

where α, β, γ are the angles which line P_1P_2 makes with the positive x, y, z axes respectively and d is given by 12.1 [see Fig. 12-1].

RELATIONSHIP BETWEEN DIRECTION COSINES

12.3 $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1$$

DIRECTION NUMBERS

Numbers L, M, N which are proportional to the direction cosines l, m, n are called *direction numbers*. The relationship between them is given by

12.4 $$l = \frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad m = \frac{M}{\sqrt{L^2 + M^2 + N^2}}, \quad n = \frac{N}{\sqrt{L^2 + M^2 + N^2}}$$

EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN STANDARD FORM

12.5 $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \text{or} \quad \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

These are also valid if l, m, n are replaced by L, M, N respectively.

EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN PARAMETRIC FORM

12.6 $$x = x_1 + lt, \quad y = y_1 + mt, \quad z = z_1 + nt$$

These are also valid if l, m, n are replaced by L, M, N respectively.

ANGLE ϕ BETWEEN TWO LINES WITH DIRECTION COSINES l_1, m_1, n_1 AND l_2, m_2, n_2

12.7 $$\cos\phi = l_1 l_2 + m_1 m_2 + n_1 n_2$$

GENERAL EQUATION OF A PLANE

12.8 $$Ax + By + Cz + D = 0 \qquad [A, B, C, D \text{ are constants}]$$

EQUATION OF PLANE PASSING THROUGH POINTS (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)

12.9 $$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

or

12.10 $$\begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix}(x - x_1) + \begin{vmatrix} z_2 - z_1 & x_2 - x_1 \\ z_3 - z_1 & x_3 - x_1 \end{vmatrix}(y - y_1) + \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}(z - z_1) = 0$$

EQUATION OF PLANE IN INTERCEPT FORM

12.11 $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, c are the intercepts on the x, y, z axes respectively.

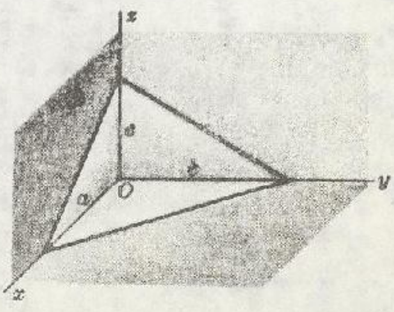

Fig. 12-2

EQUATIONS OF LINE THROUGH (x_0, y_0, z_0) AND PERPENDICULAR TO PLANE $Ax + By + Cz + D = 0$

12.12 $$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C} \quad \text{or} \quad x = x_0 + At, \;\; y = y_0 + Bt, \;\; z = z_0 + Ct$$

Note that the direction numbers for a line perpendicular to the plane $Ax + By + Cz + D = 0$ are A, B, C.

DISTANCE FROM POINT (x_0, y_0, z_0) TO PLANE $Ax + By + Cz + D = 0$

12.13 $$\frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

where the sign is chosen so that the distance is nonnegative.

NORMAL FORM FOR EQUATION OF PLANE

12.14 $$x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

where p = perpendicular distance from O to plane at P and α, β, γ are angles between OP and positive x, y, z axes.

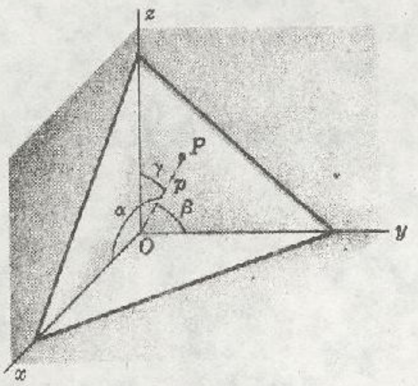

Fig. 12-3

TRANSFORMATION OF COORDINATES INVOLVING PURE TRANSLATION

12.15 $$\begin{cases} x = x' + x_0 \\ y = y' + y_0 \\ z = z' + z_0 \end{cases} \quad \text{or} \quad \begin{cases} x' = x - x_0 \\ y' = y - y_0 \\ z' = z - z_0 \end{cases}$$

where (x, y, z) are old coordinates [i.e. coordinates relative to xyz system], (x', y', z') are new coordinates [relative to $x'y'z'$ system] and (x_0, y_0, z_0) are the coordinates of the new origin O' relative to the old xyz coordinate system.

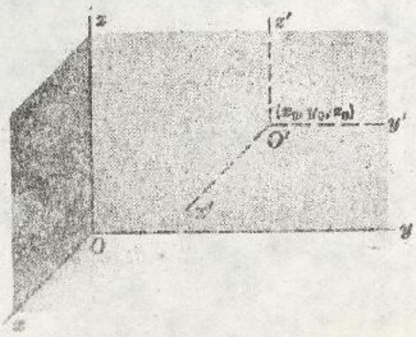

Fig. 12-4

TRANSFORMATION OF COORDINATES INVOLVING PURE ROTATION

12.16 $$\begin{cases} x = l_1x' + l_2y' + l_3z' \\ y = m_1x' + m_2y' + m_3z' \\ z = n_1x' + n_2y' + n_3z' \end{cases}$$

or $$\begin{cases} x' = l_1x + m_1y + n_1z \\ y' = l_2x + m_2y + n_2z \\ z' = l_3x + m_3y + n_3z \end{cases}$$

where the origins of the xyz and $x'y'z'$ systems are the same and l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of the x', y', z' axes relative to the x, y, z axes respectively.

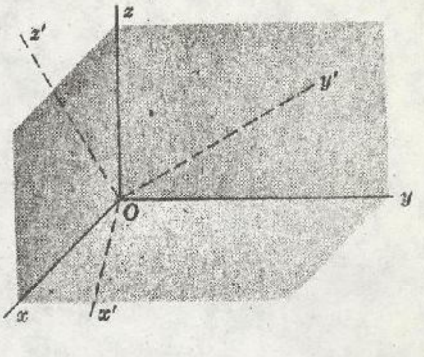

Fig. 12-5

TRANSFORMATION OF COORDINATES INVOLVING TRANSLATION AND ROTATION

12.17 $$\begin{cases} x = l_1x' + l_2y' + l_3z' + x_0 \\ y = m_1x' + m_2y' + m_3z' + y_0 \\ z = n_1x' + n_2y' + n_3z' + z_0 \end{cases}$$

or $$\begin{cases} x' = l_1(x - x_0) + m_1(y - y_0) + n_1(z - z_0) \\ y' = l_2(x - x_0) + m_2(y - y_0) + n_2(z - z_0) \\ z' = l_3(x - x_0) + m_3(y - y_0) + n_3(z - z_0) \end{cases}$$

where the origin O' of the $x'y'z'$ system has coordinates (x_0, y_0, z_0) relative to the xyz system and l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of the x', y', z' axes relative to the x, y, z axes respectively.

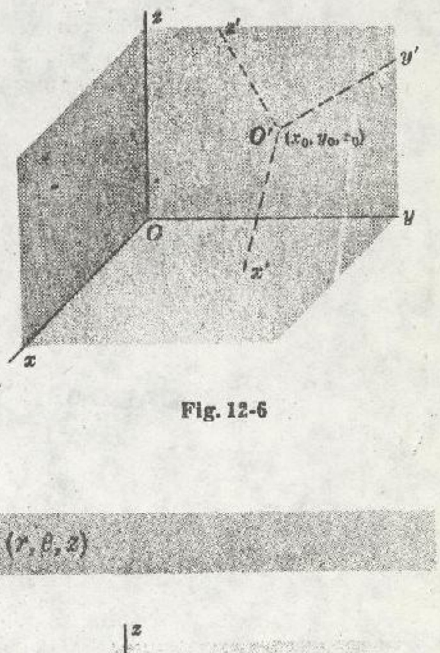

Fig. 12-6

CYLINDRICAL COORDINATES (r, θ, z)

A point P can be located by cylindrical coordinates (r, θ, z) [see Fig. 12-7] as well as rectangular coordinates (x, y, z).

The transformation between these coordinates is

12.18 $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

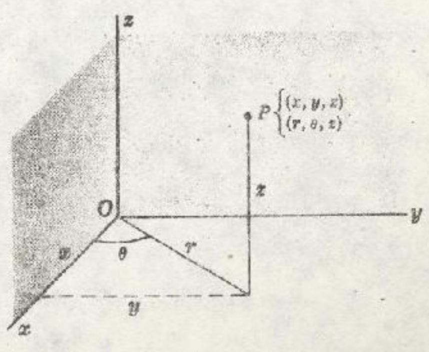

Fig. 12-7

SPHERICAL COORDINATES (r, θ, ϕ)

A point P can be located by spherical coordinates (r, θ, ϕ) [see Fig. 12-8] as well as rectangular coordinates (x, y, z).

The transformation between those coordinates is

12.19
$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases}$$

or
$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \tan^{-1}(y/x) \\ \theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right) \end{cases}$$

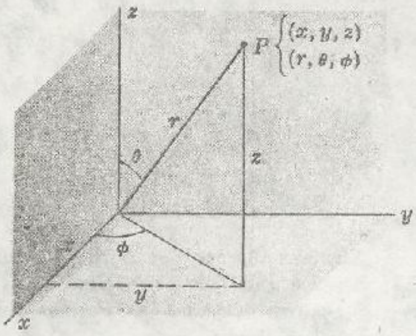

Fig. 12-8

EQUATION OF SPHERE IN RECTANGULAR COORDINATES

12.20
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2$$

where the sphere has center (x_0, y_0, z_0) and radius R.

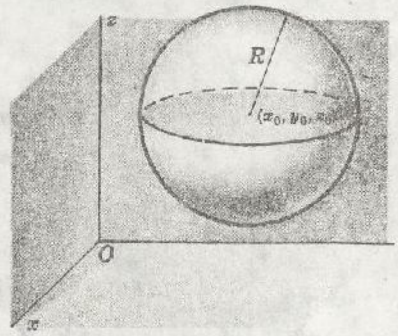

Fig. 12-9

EQUATION OF SPHERE IN CYLINDRICAL COORDINATES

12.21
$$r^2 - 2r_0 r\cos(\theta-\theta_0) + r_0^2 + (z-z_0)^2 = R^2$$

where the sphere has center (r_0, θ_0, z_0) in cylindrical coordinates and radius R.

If the center is at the origin the equation is

12.22
$$r^2 + z^2 = R^2$$

EQUATION OF SPHERE IN SPHERICAL COORDINATES

12.23
$$r^2 + r_0^2 - 2r_0 r \sin\theta \sin\theta_0 \cos(\phi-\phi_0) = R^2$$

where the sphere has center (r_0, θ_0, ϕ_0) in spherical coordinates and radius R.

If the center is at the origin the equation is

12.24
$$r = R$$

EQUATION OF ELLIPSOID WITH CENTER (x_0, y_0, z_0) AND SEMI-AXES a, b, c

12.25 $$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

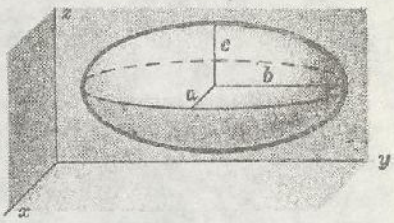

Fig. 12-10

ELLIPTIC CYLINDER WITH AXIS AS z AXIS

12.26 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a, b are semi-axes of elliptic cross section.

If $b = a$ it becomes a circular cylinder of radius a.

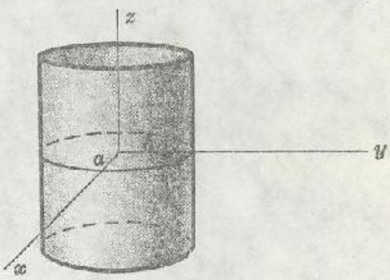

Fig. 12-11

ELLIPTIC CONE WITH AXIS AS z AXIS

12.27 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

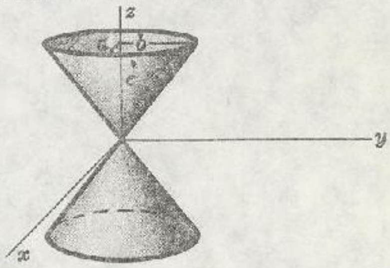

Fig. 12-12

HYPERBOLOID OF ONE SHEET

12.28 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

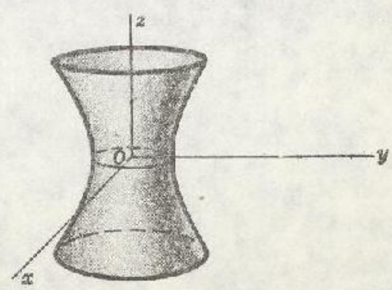

Fig. 12-13

HYPERBOLOID OF TWO SHEETS

12.29 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Note orientation of axes in Fig. 12-14.

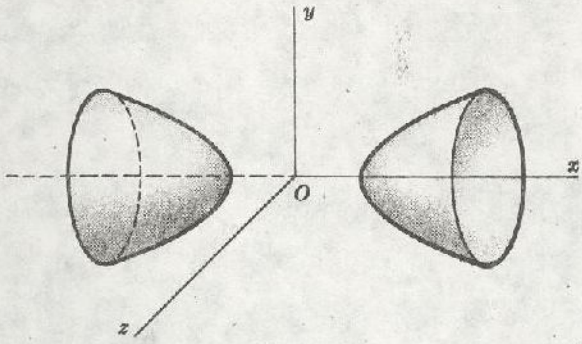

Fig. 12-14

ELLIPTIC PARABOLOID

12.30 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

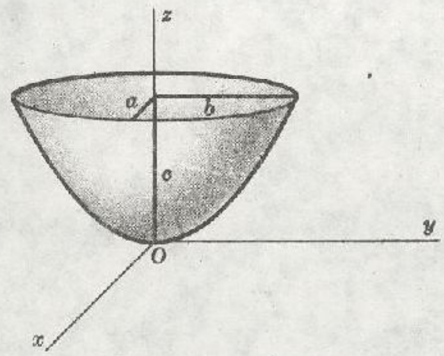

Fig. 12-15

HYPERBOLIC PARABOLOID

12.31 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Note orientation of axes in Fig. 12-16.

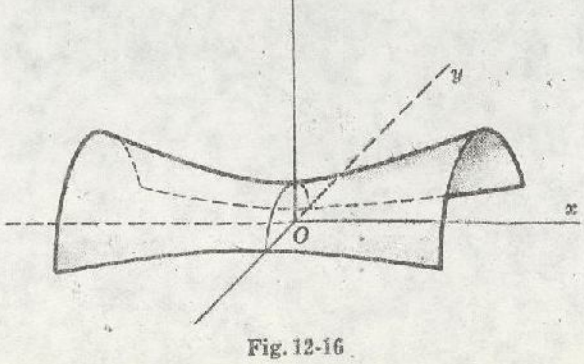

Fig. 12-16

13 DERIVATIVES

DEFINITION OF A DERIVATIVE

If $y = f(x)$, the derivative of y or $f(x)$ with respect to x is defined as

13.1 $$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

where $h = \Delta x$. The derivative is also denoted by y', df/dx or $f'(x)$. The process of taking a derivative is called *differentiation*.

GENERAL RULES OF DIFFERENTIATION

In the following, u, v, w are functions of x; a, b, c, n are constants [restricted if indicated]; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ is the natural logarithm of u [i.e. the logarithm to the base e] where it is assumed that $u > 0$ and all angles are in radians.

13.2 $$\frac{d}{dx}(c) = 0$$

13.3 $$\frac{d}{dx}(cx) = c$$

13.4 $$\frac{d}{dx}(cx^n) = ncx^{n-1}$$

13.5 $$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$$

13.6 $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

13.7 $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

13.8 $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

13.9 $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

13.10 $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

13.11 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$

13.12 $$\frac{du}{dx} = \frac{1}{dx/du}$$

13.13 $$\frac{dy}{dx} = \frac{dy/du}{dx/du}$$

DERIVATIVES OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

13.14 $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

13.15 $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

13.16 $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

13.17 $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$

13.18 $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$

13.19 $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$

13.20 $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right]$

13.21 $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad [0 < \cos^{-1} u < \pi]$

13.22 $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right]$

13.23 $\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx} \qquad [0 < \cot^{-1} u < \pi]$

13.24 $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\pm 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \left[\begin{array}{l} + \text{ if } 0 < \sec^{-1} u < \pi/2 \\ - \text{ if } \pi/2 < \sec^{-1} u < \pi \end{array}\right]$

13.25 $\frac{d}{dx}\csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \left[\begin{array}{l} - \text{ if } 0 < \csc^{-1} u < \pi/2 \\ + \text{ if } -\pi/2 < \csc^{-1} u < 0 \end{array}\right]$

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

13.26 $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \qquad a \neq 0, 1$

13.27 $\frac{d}{dx}\ln u = \frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}$

13.28 $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$

13.29 $\frac{d}{dx}e^u = e^u \frac{du}{dx}$

13.30 $\frac{d}{dx}u^v = \frac{d}{dx}e^{v\ln u} = e^{v\ln u}\frac{d}{dx}[v \ln u] = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$

DERIVATIVES OF HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

13.31 $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$

13.32 $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$

13.33 $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$

13.34 $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}$

13.35 $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

13.36 $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

13.37 $\dfrac{d}{dx}\sinh^{-1} u \;=\; \dfrac{1}{\sqrt{u^2+1}}\dfrac{du}{dx}$

13.38 $\dfrac{d}{dx}\cosh^{-1} u \;=\; \dfrac{\pm 1}{\sqrt{u^2-1}}\dfrac{du}{dx}$ $\left[\begin{array}{l} + \text{ if } \cosh^{-1} u > 0,\ u > 1 \\ - \text{ if } \cosh^{-1} u < 0,\ u > 1 \end{array}\right]$

13.39 $\dfrac{d}{dx}\tanh^{-1} u \;=\; \dfrac{1}{1-u^2}\dfrac{du}{dx}$ $[-1 < u < 1]$

13.40 $\dfrac{d}{dx}\coth^{-1} u \;=\; \dfrac{1}{1-u^2}\dfrac{du}{dx}$ $[u > 1 \text{ or } u < -1]$

13.41 $\dfrac{d}{dx}\operatorname{sech}^{-1} u \;=\; \dfrac{\mp 1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$ $\left[\begin{array}{l} - \text{ if } \operatorname{sech}^{-1} u > 0,\ 0 < u < 1 \\ + \text{ if } \operatorname{sech}^{-1} u < 0,\ 0 < u < 1 \end{array}\right]$

13.42 $\dfrac{d}{dx}\operatorname{csch}^{-1} u \;=\; \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx} \;=\; \dfrac{\mp 1}{u\sqrt{1+u^2}}\dfrac{du}{dx}$ $[- \text{ if } u > 0,\ + \text{ if } u < 0]$

HIGHER DERIVATIVES

The second, third and higher derivatives are defined as follows.

13.43 Second derivative $= \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} = f''(x) = y''$

13.44 Third derivative $= \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^3y}{dx^3} = f'''(x) = y'''$

13.45 nth derivative $= \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right) = \dfrac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$

LEIBNITZ'S RULE FOR HIGHER DERIVATIVES OF PRODUCTS

Let D^p stand for the operator $\dfrac{d^p}{dx^p}$ so that $D^pu = \dfrac{d^pu}{dx^p}$ = the pth derivative of u. Then

13.46 $$D^n(uv) \;=\; uD^nv + \binom{n}{1}(Du)(D^{n-1}v) + \binom{n}{2}(D^2u)(D^{n-2}v) + \cdots + vD^nu$$

where $\binom{n}{1}, \binom{n}{2}, \ldots$ are the binomial coefficients [page 3].

As special cases we have

13.47 $$\frac{d^2}{dx^2}(uv) \;=\; u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

13.48 $$\frac{d^3}{dx^3}(uv) \;=\; u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

DIFFERENTIALS

Let $y = f(x)$ and $\Delta y = f(x+\Delta x) - f(x)$. Then

13.49 $$\frac{\Delta y}{\Delta x} \;=\; \frac{f(x+\Delta x) - f(x)}{\Delta x} \;=\; f'(x) + \epsilon \;=\; \frac{dy}{dx} + \epsilon$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Thus

13.50 $$\Delta y \;=\; f'(x)\,\Delta x + \epsilon\,\Delta x$$

If we call $\Delta x = dx$ the differential of x, then we define the differential of y to be

13.51 $$dy \;=\; f'(x)\,dx$$

RULES FOR DIFFERENTIALS

The rules for differentials are exactly analogous to those for derivatives. As examples we observe that

13.52 $$d(u \pm v \pm w \pm \cdots) = du \pm dv \pm dw \pm \cdots$$

13.53 $$d(uv) = u\,dv + v\,du$$

13.54 $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

13.55 $$d(u^n) = nu^{n-1}\,du$$

13.56 $$d(\sin u) = \cos u\,du$$

13.57 $$d(\cos u) = -\sin u\,du$$

PARTIAL DERIVATIVES

Let $f(x, y)$ be a function of the two variables x and y. Then we define the partial derivative of $f(x, y)$ with respect to x, keeping y constant, to be

13.58 $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Similarly the partial derivative of $f(x, y)$ with respect to y, keeping x constant, is defined to be

13.59 $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial derivatives of higher order can be defined as follows.

13.60 $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

13.61 $$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The results in 13.61 will be equal if the function and its partial derivatives are continuous, i.e. in such case the order of differentiation makes no difference.

The differential of $f(x, y)$ is defined as

13.62 $$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

where $dx = \Delta x$ and $dy = \Delta y$.

Extension to functions of more than two variables are exactly analogous.

14 INDEFINITE INTEGRALS

DEFINITION OF AN INDEFINITE INTEGRAL

If $\dfrac{dy}{dx} = f(x)$, then y is the function whose derivative is $f(x)$ and is called the *anti-derivative* of $f(x)$ or the *indefinite integral* of $f(x)$, denoted by $\int f(x)\,dx$. Similarly if $y = \int f(u)\,du$, then $\dfrac{dy}{du} = f(u)$. Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant.

For the definition of a definite integral, see page 94. The process of finding an integral is called *integration.*

GENERAL RULES OF INTEGRATION

In the following, u, v, w are functions of x; a, b, p, q, n any constants, restricted if indicated; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ denotes the natural logarithm of u where it is assumed that $u > 0$ [in general, to extend formulas to cases where $u < 0$ as well, replace $\ln u$ by $\ln|u|$]; all angles are in radians; all constants of integration are omitted but implied.

14.1 $$\int a\,dx = ax$$

14.2 $$\int af(x)\,dx = a\int f(x)\,dx$$

14.3 $$\int (u \pm v \pm w \pm \cdots)\,dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \cdots$$

14.4 $$\int u\,dv = uv - \int v\,du \qquad \text{[Integration by parts]}$$

For generalized integration by parts, see 14.48.

14.5 $$\int f(ax)\,dx = \frac{1}{a}\int f(u)\,du$$

14.6 $$\int F\{f(x)\}\,dx = \int F(u)\frac{dx}{du}\,du = \int \frac{F(u)}{f'(x)}\,du \quad \text{where } u = f(x)$$

14.7 $$\int u^n\,du = \frac{u^{n+1}}{n+1}, \quad n \neq -1 \qquad \text{[For } n = -1, \text{ see 14.8]}$$

14.8 $$\int \frac{du}{u} = \ln u \quad \text{if } u > 0 \text{ or } \ln(-u) \text{ if } u < 0$$
$$= \ln|u|$$

14.9 $$\int e^u\,du = e^u$$

14.10 $$\int a^u\,du = \int e^{u\ln a}\,du = \frac{e^{u\ln a}}{\ln a} = \frac{a^u}{\ln a}, \quad a > 0,\ a \neq 1$$

14.11 $\int \sin u\,du = -\cos u$

14.12 $\int \cos u\,du = \sin u$

14.13 $\int \tan u\,du = \ln \sec u = -\ln \cos u$

14.14 $\int \cot u\,du = \ln \sin u$

14.15 $\int \sec u\,du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2}+\frac{\pi}{4}\right)$

14.16 $\int \csc u\,du = \ln(\csc u - \cot u) = \ln \tan \frac{u}{2}$

14.17 $\int \sec^2 u\,du = \tan u$

14.18 $\int \csc^2 u\,du = -\cot u$

14.19 $\int \tan^2 u\,du = \tan u - u$

14.20 $\int \cot^2 u\,du = -\cot u - u$

14.21 $\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \frac{1}{2}(u - \sin u \cos u)$

14.22 $\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{1}{2}(u + \sin u \cos u)$

14.23 $\int \sec u \tan u\,du = \sec u$

14.24 $\int \csc u \cot u\,du = -\csc u$

14.25 $\int \sinh u\,du = \cosh u$

14.26 $\int \cosh u\,du = \sinh u$

14.27 $\int \tanh u\,du = \ln \cosh u$

14.28 $\int \coth u\,du = \ln \sinh u$

14.29 $\int \operatorname{sech} u\,du = \sin^{-1}(\tanh u)$ or $2\tan^{-1} e^u$

14.30 $\int \operatorname{csch} u\,du = \ln \tanh \frac{u}{2}$ or $-\coth^{-1} e^u$

14.31 $\int \operatorname{sech}^2 u\,du = \tanh u$

14.32 $\int \operatorname{csch}^2 u\,du = -\coth u$

14.33 $\int \tanh^2 u\,du = u - \tanh u$

14.34 $\displaystyle\int \coth^2 u\,du = u - \coth u$

14.35 $\displaystyle\int \sinh^2 u\,du = \frac{\sinh 2u}{4} - \frac{u}{2} = \tfrac{1}{2}(\sinh u \cosh u - u)$

14.36 $\displaystyle\int \cosh^2 u\,du = \frac{\sinh 2u}{4} + \frac{u}{2} = \tfrac{1}{2}(\sinh u \cosh u + u)$

14.37 $\displaystyle\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u$

14.38 $\displaystyle\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u$

14.39 $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

14.40 $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right) = -\frac{1}{a}\coth^{-1}\frac{u}{a} \qquad u^2 > a^2$

14.41 $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left(\frac{a+u}{a-u}\right) = \frac{1}{a}\tanh^{-1}\frac{u}{a} \qquad u^2 < a^2$

14.42 $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$

14.43 $\displaystyle\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u+\sqrt{u^2+a^2}) \quad \text{or} \quad \sinh^{-1}\frac{u}{a}$

14.44 $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln(u+\sqrt{u^2-a^2})$

14.45 $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|$

14.46 $\displaystyle\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{u^2+a^2}}{u}\right)$

14.47 $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{u}\right)$

14.48 $\displaystyle\int f^{(n)}g\,dx = f^{(n-1)}g - f^{(n-2)}g' + f^{(n-3)}g'' - \cdots (-1)^n \int fg^{(n)}\,dx$

This is called *generalized integration* by parts.

IMPORTANT TRANSFORMATIONS

Often in practice an integral can be simplified by using an appropriate transformation or substitution and formula 14.6, page 57. The following list gives some transformations and their effects.

14.49 $\displaystyle\int F(ax+b)\,dx = \frac{1}{a}\int F(u)\,du \qquad \text{where } u = ax+b$

14.50 $\displaystyle\int F(\sqrt{ax+b})\,dx = \frac{2}{a}\int u\,F(u)\,du \qquad \text{where } u = \sqrt{ax+b}$

14.51 $\displaystyle\int F(\sqrt[n]{ax+b})\,dx = \frac{n}{a}\int u^{n-1}F(u)\,du \qquad \text{where } u = \sqrt[n]{ax+b}$

14.52 $\displaystyle\int F(\sqrt{a^2-x^2})\,dx = a\int F(a\cos u)\cos u\,du \qquad \text{where } x = a\sin u$

14.53 $\displaystyle\int F(\sqrt{x^2+a^2})\,dx = a\int F(a\sec u)\sec^2 u\,du \qquad \text{where } x = a\tan u$

14.54 $\displaystyle\int F(\sqrt{x^2-a^2})\,dx = a\int F(a\tan u)\sec u\tan u\,du$ where $x = a\sec u$

14.55 $\displaystyle\int F(e^{ax})\,dx = \frac{1}{a}\int \frac{F(u)}{u}\,du$ where $u = e^{ax}$

14.56 $\displaystyle\int F(\ln x)\,dx = \int F(u)\,e^u\,du$ where $u = \ln x$

14.57 $\displaystyle\int F\left(\sin^{-1}\frac{x}{a}\right)dx = a\int F(u)\cos u\,du$ where $u = \sin^{-1}\dfrac{x}{a}$

Similar results apply for other inverse trigonometric functions.

14.58 $\displaystyle\int F(\sin x, \cos x)\,dx = 2\int F\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}$ where $u = \tan\dfrac{x}{2}$

SPECIAL INTEGRALS

Pages 60 through 93 provide a table of integrals classified under special types. The remarks given on page 57 apply here as well. It is assumed in all cases that division by zero is excluded.

INTEGRALS INVOLVING $ax+b$

14.59 $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b)$

14.60 $\displaystyle\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln(ax+b)$

14.61 $\displaystyle\int \frac{x^2\,dx}{ax+b} = \frac{(ax+b)^2}{2a^3} - \frac{2b(ax+b)}{a^3} + \frac{b^2}{a^3}\ln(ax+b)$

14.62 $\displaystyle\int \frac{x^3\,dx}{ax+b} = \frac{(ax+b)^3}{3a^4} - \frac{3b(ax+b)^2}{2a^4} + \frac{3b^2(ax+b)}{a^4} - \frac{b^3}{a^4}\ln(ax+b)$

14.63 $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left(\frac{x}{ax+b}\right)$

14.64 $\displaystyle\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left(\frac{ax+b}{x}\right)$

14.65 $\displaystyle\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3}\ln\left(\frac{x}{ax+b}\right)$

14.66 $\displaystyle\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)}$

14.67 $\displaystyle\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln(ax+b)$

14.68 $\displaystyle\int \frac{x^2\,dx}{(ax+b)^2} = \frac{ax+b}{a^3} - \frac{b^2}{a^3(ax+b)} - \frac{2b}{a^3}\ln(ax+b)$

14.69 $\displaystyle\int \frac{x^3\,dx}{(ax+b)^2} = \frac{(ax+b)^2}{2a^4} - \frac{3b(ax+b)}{a^4} + \frac{b^3}{a^4(ax+b)} + \frac{3b^2}{a^4}\ln(ax+b)$

14.70 $\displaystyle\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\ln\left(\frac{x}{ax+b}\right)$

14.71 $\displaystyle\int \frac{dx}{x^2(ax+b)^2} = \frac{-a}{b^2(ax+b)} - \frac{1}{b^2x} + \frac{2a}{b^3}\ln\left(\frac{ax+b}{x}\right)$

14.72 $\displaystyle\int \frac{dx}{x^3(ax+b)^2} = -\frac{(ax+b)^2}{2b^4x^2} + \frac{3a(ax+b)}{b^4x} - \frac{a^3x}{b^4(ax+b)} - \frac{3a^2}{b^4}\ln\left(\frac{ax+b}{x}\right)$

14.73 $\displaystyle\int \frac{dx}{(ax+b)^3} = \frac{-1}{2(ax+b)^2}$

14.74 $\displaystyle\int \frac{x\,dx}{(ax+b)^3} = \frac{-1}{a^2(ax+b)} + \frac{b}{2a^2(ax+b)^2}$

14.75 $\displaystyle\int \frac{x^2\,dx}{(ax+b)^3} = \frac{2b}{a^3(ax+b)} - \frac{b^2}{2a^3(ax+b)^2} + \frac{1}{a^3}\ln(ax+b)$

14.76 $\displaystyle\int \frac{x^3\,dx}{(ax+b)^3} = \frac{x}{a^3} - \frac{3b^2}{a^4(ax+b)} + \frac{b^3}{2a^4(ax+b)^2} - \frac{3b}{a^4}\ln(ax+b)$

14.77 $\displaystyle\int \frac{dx}{x(ax+b)^3} = \frac{a^2x^2}{2b^3(ax+b)^2} - \frac{2ax}{b^3(ax+b)} - \frac{1}{b^3}\ln\left(\frac{ax+b}{x}\right)$

14.78 $\displaystyle\int \frac{dx}{x^2(ax+b)^3} = \frac{-a}{2b^2(ax+b)^2} - \frac{2a}{b^3(ax+b)} - \frac{1}{b^3x} + \frac{3a}{b^4}\ln\left(\frac{ax+b}{x}\right)$

14.79 $\displaystyle\int \frac{dx}{x^3(ax+b)^3} = \frac{a^4x^2}{2b^5(ax+b)^2} - \frac{4a^3x}{b^5(ax+b)} - \frac{(ax+b)^2}{2b^5x^2} - \frac{6a^2}{b^5}\ln\left(\frac{ax+b}{x}\right)$

14.80 $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}.$ If $n=-1$, see 14.59.

14.81 $\displaystyle\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}, \quad n \neq -1, -2$

If $n = -1, -2$, see 14.60, 14.67.

14.82 $\displaystyle\int x^2(ax+b)^n\,dx = \frac{(ax+b)^{n+3}}{(n+3)a^3} - \frac{2b(ax+b)^{n+2}}{(n+2)a^3} + \frac{b^2(ax+b)^{n+1}}{(n+1)a^3}$

If $n = -1, -2, -3$, see 14.61, 14.68, 14.75.

14.83
$$\int x^m(ax+b)^n\,dx = \begin{cases} \dfrac{x^{m+1}(ax+b)^n}{m+n+1} + \dfrac{nb}{m+n+1}\displaystyle\int x^m(ax+b)^{n-1}\,dx \\[2ex] \dfrac{x^m(ax+b)^{n+1}}{(m+n+1)a} - \dfrac{mb}{(m+n+1)a}\displaystyle\int x^{m-1}(ax+b)^n\,dx \\[2ex] \dfrac{-x^{m+1}(ax+b)^{n+1}}{(n+1)b} + \dfrac{m+n+2}{(n+1)b}\displaystyle\int x^m(ax+b)^{n+1}\,dx \end{cases}$$

INTEGRALS INVOLVING $\sqrt{ax+b}$

14.84 $\displaystyle\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}$

14.85 $\displaystyle\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$

14.86 $\displaystyle\int \frac{x^2\,dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2 - 4abx + 8b^2)}{15a^3}\sqrt{ax+b}$

14.87
$$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{b}}\ln\left(\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right) \\[2ex] \dfrac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\dfrac{ax+b}{-b}} \end{cases}$$

14.88 $\displaystyle\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$ [See 14.87]

14.89 $$\int \sqrt{ax+b}\,dx = \frac{2\sqrt{(ax+b)^3}}{3a}$$

14.90 $$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$$

14.91 $$\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}\sqrt{(ax+b)^3}$$

14.92 $$\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int\frac{dx}{x\sqrt{ax+b}}$$ [See 14.87]

14.93 $$\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int\frac{dx}{x\sqrt{ax+b}}$$ [See 14.87]

14.94 $$\int \frac{x^m}{\sqrt{ax+b}}\,dx = \frac{2x^m\sqrt{ax+b}}{(2m+1)a} - \frac{2mb}{(2m+1)a}\int\frac{x^{m-1}}{\sqrt{ax+b}}\,dx$$

14.95 $$\int \frac{dx}{x^m\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{(2m-2)b}\int\frac{dx}{x^{m-1}\sqrt{ax+b}}$$

14.96 $$\int x^m\sqrt{ax+b}\,dx = \frac{2x^m}{(2m+3)a}(ax+b)^{3/2} - \frac{2mb}{(2m+3)a}\int x^{m-1}\sqrt{ax+b}\,dx$$

14.97 $$\int \frac{\sqrt{ax+b}}{x^m}\,dx = -\frac{\sqrt{ax+b}}{(m-1)x^{m-1}} + \frac{a}{2(m-1)}\int\frac{dx}{x^{m-1}\sqrt{ax+b}}$$

14.98 $$\int \frac{\sqrt{ax+b}}{x^m}\,dx = \frac{-(ax+b)^{3/2}}{(m-1)bx^{m-1}} - \frac{(2m-5)a}{(2m-2)b}\int\frac{\sqrt{ax+b}}{x^{m-1}}\,dx$$

14.99 $$\int (ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+2)/2}}{a(m+2)}$$

14.100 $$\int x(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+4)/2}}{a^2(m+4)} - \frac{2b(ax+b)^{(m+2)/2}}{a^2(m+2)}$$

14.101 $$\int x^2(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+6)/2}}{a^3(m+6)} - \frac{4b(ax+b)^{(m+4)/2}}{a^3(m+4)} + \frac{2b^2(ax+b)^{(m+2)/2}}{a^3(m+2)}$$

14.102 $$\int \frac{(ax+b)^{m/2}}{x}\,dx = \frac{2(ax+b)^{m/2}}{m} + b\int\frac{(ax+b)^{(m-2)/2}}{x}\,dx$$

14.103 $$\int \frac{(ax+b)^{m/2}}{x^2}\,dx = -\frac{(ax+b)^{(m+2)/2}}{bx} + \frac{ma}{2b}\int\frac{(ax+b)^{m/2}}{x}\,dx$$

14.104 $$\int \frac{dx}{x(ax+b)^{m/2}} = \frac{2}{(m-2)b(ax+b)^{(m-2)/2}} + \frac{1}{b}\int\frac{dx}{x(ax+b)^{(m-2)/2}}$$

INTEGRALS INVOLVING $ax+b$ AND $px+q$

14.105 $$\int \frac{dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)$$

14.106 $$\int \frac{x\,dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\left\{\frac{b}{a}\ln(ax+b) - \frac{q}{p}\ln(px+q)\right\}$$

14.107 $$\int \frac{dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{1}{ax+b} + \frac{p}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)\right\}$$

14.108 $$\int \frac{x\,dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{q}{bp-aq}\ln\left(\frac{ax+b}{px+q}\right) - \frac{b}{a(ax+b)}\right\}$$

14.109 $$\int \frac{x^2\,dx}{(ax+b)^2(px+q)} = \frac{b^2}{(bp-aq)a^2(ax+b)} + \frac{1}{(bp-aq)^2}\left\{\frac{q^2}{p}\ln(px+q) + \frac{b(bp-2aq)}{a^2}\ln(ax+b)\right\}$$

14.110 $$\int \frac{dx}{(ax+b)^m(px+q)^n} = \frac{-1}{(n-1)(bp-aq)}\left\{\frac{1}{(ax+b)^{m-1}(px+q)^{n-1}} + a(m+n-2)\int \frac{dx}{(ax+b)^m(px+q)^{n-1}}\right\}$$

14.111 $$\int \frac{ax+b}{px+q}\,dx = \frac{ax}{p} + \frac{bp-aq}{p^2}\ln(px+q)$$

14.112 $$\int \frac{(ax+b)^m}{(px+q)^n}\,dx = \begin{cases} \dfrac{-1}{(n-1)(bp-aq)}\left\{\dfrac{(ax+b)^{m+1}}{(px+q)^{n-1}} + (n-m-2)a\displaystyle\int \dfrac{(ax+b)^m}{(px+q)^{n-1}}\,dx\right\} \\ \dfrac{-1}{(n-m-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} + m(bp-aq)\displaystyle\int \dfrac{(ax+b)^{m-1}}{(px+q)^n}\,dx\right\} \\ \dfrac{-1}{(n-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} - ma\displaystyle\int \dfrac{(ax+b)^{m-1}}{(px+q)^{n-1}}\,dx\right\} \end{cases}$$

INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $px+q$

14.113 $$\int \frac{px+q}{\sqrt{ax+b}}\,dx = \frac{2(apx+3aq-2bp)}{3a^2}\sqrt{ax+b}$$

14.114 $$\int \frac{dx}{(px+q)\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{bp-aq}\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2}{\sqrt{aq-bp}\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

14.115 $$\int \frac{\sqrt{ax+b}}{px+q}\,dx = \begin{cases} \dfrac{2\sqrt{ax+b}}{p} + \dfrac{\sqrt{bp-aq}}{p\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2\sqrt{ax+b}}{p} - \dfrac{2\sqrt{aq-bp}}{p\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

14.116 $$\int (px+q)^n\sqrt{ax+b}\,dx = \frac{2(px+q)^{n+1}\sqrt{ax+b}}{(2n+3)p} + \frac{bp-aq}{(2n+3)p}\int \frac{(px+q)^n}{\sqrt{ax+b}}\,dx$$

14.117 $$\int \frac{dx}{(px+q)^n\sqrt{ax+b}} = \frac{\sqrt{ax+b}}{(n-1)(aq-bp)(px+q)^{n-1}} + \frac{(2n-3)a}{2(n-1)(aq-bp)}\int \frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

14.118 $$\int \frac{(px+q)^n}{\sqrt{ax+b}}\,dx = \frac{2(px+q)^n\sqrt{ax+b}}{(2n+1)a} + \frac{2n(aq-bp)}{(2n+1)a}\int \frac{(px+q)^{n-1}\,dx}{\sqrt{ax+b}}$$

14.119 $$\int \frac{\sqrt{ax+b}}{(px+q)^n}\,dx = \frac{-\sqrt{ax+b}}{(n-1)p(px+q)^{n-1}} + \frac{a}{2(n-1)p}\int \frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $\sqrt{px+q}$

14.120 $$\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \begin{cases} \dfrac{2}{\sqrt{ap}}\ln\left(\sqrt{a(px+q)}+\sqrt{p(ax+b)}\right) \\ \dfrac{2}{\sqrt{-ap}}\tan^{-1}\sqrt{\dfrac{-p(ax+b)}{a(px+q)}} \end{cases}$$

14.121 $$\int \frac{x\,dx}{\sqrt{(ax+b)(px+q)}} = \frac{\sqrt{(ax+b)(px+q)}}{ap} - \frac{bp+aq}{2ap}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

14.122 $$\int \sqrt{(ax+b)(px+q)}\,dx = \frac{2apx+bp+aq}{4ap}\sqrt{(ax+b)(px+q)} - \frac{(bp-aq)^2}{8ap}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

14.123 $$\int \sqrt{\frac{px+q}{ax+b}}\,dx = \frac{\sqrt{(ax+b)(px+q)}}{a} + \frac{aq-bp}{2a}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

14.124 $$\int \frac{dx}{(px+q)\sqrt{(ax+b)(px+q)}} = \frac{2\sqrt{ax+b}}{(aq-bp)\sqrt{px+q}}$$

INTEGRALS INVOLVING $x^2 + a^2$

14.125 $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

14.126 $$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2)$$

14.127 $$\int \frac{x^2\,dx}{x^2+a^2} = x - a\tan^{-1}\frac{x}{a}$$

14.128 $$\int \frac{x^3\,dx}{x^2+a^2} = \frac{x^2}{2} - \frac{a^2}{2}\ln(x^2+a^2)$$

14.129 $$\int \frac{dx}{x(x^2+a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

14.130 $$\int \frac{dx}{x^2(x^2+a^2)} = -\frac{1}{a^2x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a}$$

14.131 $$\int \frac{dx}{x^3(x^2+a^2)} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

14.132 $$\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

14.133 $$\int \frac{x\,dx}{(x^2+a^2)^2} = \frac{-1}{2(x^2+a^2)}$$

14.134 $$\int \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{-x}{2(x^2+a^2)} + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

14.135 $$\int \frac{x^3\,dx}{(x^2+a^2)^2} = \frac{a^2}{2(x^2+a^2)} + \frac{1}{2}\ln(x^2+a^2)$$

14.136 $$\int \frac{dx}{x(x^2+a^2)^2} = \frac{1}{2a^2(x^2+a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

14.137 $$\int \frac{dx}{x^2(x^2+a^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(x^2+a^2)} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a}$$

14.138 $$\int \frac{dx}{x^3(x^2+a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2+a^2)} - \frac{1}{a^6}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

14.139 $$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$$

14.140 $$\int \frac{x\,dx}{(x^2+a^2)^n} = \frac{-1}{2(n-1)(x^2+a^2)^{n-1}}$$

14.141 $$\int \frac{dx}{x(x^2+a^2)^n} = \frac{1}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(x^2+a^2)^{n-1}}$$

14.142 $$\int \frac{x^m\,dx}{(x^2+a^2)^n} = \int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n-1}} - a^2\int \frac{x^{m-2}\,dx}{(x^2+a^2)^n}$$

14.143 $$\int \frac{dx}{x^m(x^2+a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^m(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2+a^2)^n}$$

INTEGRALS INVOLVING $x^2 - a^2$, $x^2 > a^2$

14.144 $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) \quad \text{or} \quad -\frac{1}{a}\coth^{-1}\frac{x}{a}$$

14.145 $$\int \frac{x\,dx}{x^2 - a^2} = \frac{1}{2}\ln(x^2 - a^2)$$

14.146 $$\int \frac{x^2\,dx}{x^2 - a^2} = x + \frac{a}{2}\ln\left(\frac{x-a}{x+a}\right)$$

14.147 $$\int \frac{x^3\,dx}{x^2 - a^2} = \frac{x^2}{2} + \frac{a^2}{2}\ln(x^2 - a^2)$$

14.148 $$\int \frac{dx}{x(x^2 - a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2 - a^2}{x^2}\right)$$

14.149 $$\int \frac{dx}{x^2(x^2 - a^2)} = \frac{1}{a^2 x} + \frac{1}{2a^3}\ln\left(\frac{x-a}{x+a}\right)$$

14.150 $$\int \frac{dx}{x^3(x^2 - a^2)} = \frac{1}{2a^2 x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2 - a^2}\right)$$

14.151 $$\int \frac{dx}{(x^2 - a^2)^2} = \frac{-x}{2a^2(x^2 - a^2)} - \frac{1}{4a^3}\ln\left(\frac{x-a}{x+a}\right)$$

14.152 $$\int \frac{x\,dx}{(x^2 - a^2)^2} = \frac{-1}{2(x^2 - a^2)}$$

14.153 $$\int \frac{x^2\,dx}{(x^2 - a^2)^2} = \frac{-x}{2(x^2 - a^2)} + \frac{1}{4a}\ln\left(\frac{x-a}{x+a}\right)$$

14.154 $$\int \frac{x^3\,dx}{(x^2 - a^2)^2} = \frac{-a^2}{2(x^2 - a^2)} + \frac{1}{2}\ln(x^2 - a^2)$$

14.155 $$\int \frac{dx}{x(x^2 - a^2)^2} = \frac{-1}{2a^2(x^2 - a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2 - a^2}\right)$$

14.156 $$\int \frac{dx}{x^2(x^2 - a^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(x^2 - a^2)} - \frac{3}{4a^5}\ln\left(\frac{x-a}{x+a}\right)$$

14.157 $$\int \frac{dx}{x^3(x^2 - a^2)^2} = -\frac{1}{2a^4 x^2} - \frac{1}{2a^4(x^2 - a^2)} + \frac{1}{a^6}\ln\left(\frac{x^2}{x^2 - a^2}\right)$$

14.158 $$\int \frac{dx}{(x^2 - a^2)^n} = \frac{-x}{2(n-1)a^2(x^2 - a^2)^{n-1}} - \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2 - a^2)^{n-1}}$$

14.159 $$\int \frac{x\,dx}{(x^2 - a^2)^n} = \frac{-1}{2(n-1)(x^2 - a^2)^{n-1}}$$

14.160 $$\int \frac{dx}{x(x^2 - a^2)^n} = \frac{-1}{2(n-1)a^2(x^2 - a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x(x^2 - a^2)^{n-1}}$$

14.161 $$\int \frac{x^m\,dx}{(x^2 - a^2)^n} = \int \frac{x^{m-2}\,dx}{(x^2 - a^2)^{n-1}} + a^2\int \frac{x^{m-2}\,dx}{(x^2 - a^2)^n}$$

14.162 $$\int \frac{dx}{x^m(x^2 - a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2 - a^2)^n} - \frac{1}{a^2}\int \frac{dx}{x^m(x^2 - a^2)^{n-1}}$$

INTEGRALS INVOLVING $a^2 - x^2$, $x^2 < a^2$

14.163 $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) \quad \text{or} \quad \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

14.164 $$\int \frac{x\,dx}{a^2 - x^2} = -\frac{1}{2}\ln(a^2 - x^2)$$

14.165 $$\int \frac{x^2\,dx}{a^2 - x^2} = -x + \frac{a}{2}\ln\left(\frac{a+x}{a-x}\right)$$

14.166 $$\int \frac{x^3\,dx}{a^2 - x^2} = -\frac{x^2}{2} - \frac{a^2}{2}\ln(a^2 - x^2)$$

14.167 $$\int \frac{dx}{x(a^2 - x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.168 $$\int \frac{dx}{x^2(a^2 - x^2)} = -\frac{1}{a^2 x} + \frac{1}{2a^3}\ln\left(\frac{a+x}{a-x}\right)$$

14.169 $$\int \frac{dx}{x^3(a^2 - x^2)} = -\frac{1}{2a^2 x^2} + \frac{1}{2a^4}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.170 $$\int \frac{dx}{(a^2 - x^2)^2} = \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{4a^3}\ln\left(\frac{a+x}{a-x}\right)$$

14.171 $$\int \frac{x\,dx}{(a^2 - x^2)^2} = \frac{1}{2(a^2 - x^2)}$$

14.172 $$\int \frac{x^2\,dx}{(a^2 - x^2)^2} = \frac{x}{2(a^2 - x^2)} - \frac{1}{4a}\ln\left(\frac{a+x}{a-x}\right)$$

14.173 $$\int \frac{x^3\,dx}{(a^2 - x^2)^2} = \frac{a^2}{2(a^2 - x^2)} + \frac{1}{2}\ln(a^2 - x^2)$$

14.174 $$\int \frac{dx}{x(a^2 - x^2)^2} = \frac{1}{2a^2(a^2 - x^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.175 $$\int \frac{dx}{x^2(a^2 - x^2)^2} = \frac{-1}{a^4 x} + \frac{x}{2a^4(a^2 - x^2)} + \frac{3}{4a^5}\ln\left(\frac{a+x}{a-x}\right)$$

14.176 $$\int \frac{dx}{x^3(a^2 - x^2)^2} = \frac{-1}{2a^4 x^2} + \frac{1}{2a^4(a^2 - x^2)} + \frac{1}{a^6}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.177 $$\int \frac{dx}{(a^2 - x^2)^n} = \frac{x}{2(n-1)a^2(a^2 - x^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(a^2 - x^2)^{n-1}}$$

14.178 $$\int \frac{x\,dx}{(a^2 - x^2)^n} = \frac{1}{2(n-1)(a^2 - x^2)^{n-1}}$$

14.179 $$\int \frac{dx}{x(a^2 - x^2)^n} = \frac{1}{2(n-1)a^2(a^2 - x^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(a^2 - x^2)^{n-1}}$$

14.180 $$\int \frac{x^m\,dx}{(a^2 - x^2)^n} = a^2 \int \frac{x^{m-2}\,dx}{(a^2 - x^2)^n} - \int \frac{x^{m-2}\,dx}{(a^2 - x^2)^{n-1}}$$

14.181 $$\int \frac{dx}{x^m(a^2 - x^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^m(a^2 - x^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x^{m-2}(a^2 - x^2)^n}$$

INTEGRALS INVOLVING $\sqrt{x^2+a^2}$

14.182 $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) \quad \text{or} \quad \sinh^{-1}\frac{x}{a}$$

14.183 $$\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}$$

14.184 $$\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$$

14.185 $$\int \frac{x^3\,dx}{\sqrt{x^2+a^2}} = \frac{(x^2+a^2)^{3/2}}{3} - a^2\sqrt{x^2+a^2}$$

14.186 $$\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.187 $$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}$$

14.188 $$\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.189 $$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$$

14.190 $$\int x\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{3/2}}{3}$$

14.191 $$\int x^2\sqrt{x^2+a^2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} - \frac{a^2x\sqrt{x^2+a^2}}{8} - \frac{a^4}{8}\ln(x+\sqrt{x^2+a^2})$$

14.192 $$\int x^3\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{5/2}}{5} - \frac{a^2(x^2+a^2)^{3/2}}{3}$$

14.193 $$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.194 $$\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln(x+\sqrt{x^2+a^2})$$

14.195 $$\int \frac{\sqrt{x^2+a^2}}{x^3}\,dx = -\frac{\sqrt{x^2+a^2}}{2x^2} - \frac{1}{2a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.196 $$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

14.197 $$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}$$

14.198 $$\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = \frac{-x}{\sqrt{x^2+a^2}} + \ln(x+\sqrt{x^2+a^2})$$

14.199 $$\int \frac{x^3\,dx}{(x^2+a^2)^{3/2}} = \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}}$$

14.200 $$\int \frac{dx}{x(x^2+a^2)^{3/2}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.201 $$\int \frac{dx}{x^2(x^2+a^2)^{3/2}} = -\frac{\sqrt{x^2+a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2+a^2}}$$

14.202 $$\int \frac{dx}{x^3(x^2+a^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{x^2+a^2}} - \frac{3}{2a^4\sqrt{x^2+a^2}} + \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.203 $$\int (x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} + \frac{3a^2x\sqrt{x^2+a^2}}{8} + \frac{3}{8}a^4 \ln\,(x+\sqrt{x^2+a^2})$$

14.204 $$\int x(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{5/2}}{5}$$

14.205 $$\int x^2(x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{5/2}}{6} - \frac{a^2x(x^2+a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2+a^2}}{16} - \frac{a^6}{16}\ln\,(x+\sqrt{x^2+a^2})$$

14.206 $$\int x^3(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{7/2}}{7} - \frac{a^2(x^2+a^2)^{5/2}}{5}$$

14.207 $$\int \frac{(x^2+a^2)^{3/2}}{x}\,dx = \frac{(x^2+a^2)^{3/2}}{3} + a^2\sqrt{x^2+a^2} - a^3\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.208 $$\int \frac{(x^2+a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2+a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2+a^2}}{2} + \frac{3}{2}a^2\ln\,(x+\sqrt{x^2+a^2})$$

14.209 $$\int \frac{(x^2+a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2+a^2)^{3/2}}{2x^2} + \frac{3}{2}\sqrt{x^2+a^2} - \frac{3}{2}a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

INTEGRALS INVOLVING $\sqrt{x^2-a^2}$

14.210 $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\,(x+\sqrt{x^2-a^2}), \qquad \int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}$$

14.211 $$\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\ln\,(x+\sqrt{x^2-a^2})$$

14.212 $$\int \frac{x^3\,dx}{\sqrt{x^2-a^2}} = \frac{(x^2-a^2)^{3/2}}{3} + a^2\sqrt{x^2-a^2}$$

14.213 $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

14.214 $$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}$$

14.215 $$\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\left|\frac{x}{a}\right|$$

14.216 $$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\,(x+\sqrt{x^2-a^2})$$

14.217 $$\int x\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{3/2}}{3}$$

14.218 $$\int x^2\sqrt{x^2-a^2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} + \frac{a^2x\sqrt{x^2-a^2}}{8} - \frac{a^4}{8}\ln\,(x+\sqrt{x^2-a^2})$$

14.219 $$\int x^3\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{5/2}}{5} + \frac{a^2(x^2-a^2)^{3/2}}{3}$$

14.220 $$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right|$$

14.221 $$\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\,(x+\sqrt{x^2-a^2})$$

14.222 $$\int \frac{\sqrt{x^2-a^2}}{x^3}\,dx = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\sec^{-1}\left|\frac{x}{a}\right|$$

14.223 $$\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}}$$

14.224 $\displaystyle\int \frac{x\,dx}{(x^2-a^2)^{3/2}} = \frac{-1}{\sqrt{x^2-a^2}}$

14.225 $\displaystyle\int \frac{x^2\,dx}{(x^2-a^2)^{3/2}} = -\frac{x}{\sqrt{x^2-a^2}} + \ln(x+\sqrt{x^2-a^2})$

14.226 $\displaystyle\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}} = \sqrt{x^2-a^2} - \frac{a^2}{\sqrt{x^2-a^2}}$

14.227 $\displaystyle\int \frac{dx}{x(x^2-a^2)^{3/2}} = \frac{-1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\sec^{-1}\left|\frac{x}{a}\right|$

14.228 $\displaystyle\int \frac{dx}{x^2(x^2-a^2)^{3/2}} = -\frac{\sqrt{x^2-a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2-a^2}}$

14.229 $\displaystyle\int \frac{dx}{x^3(x^2-a^2)^{3/2}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}} - \frac{3}{2a^5}\sec^{-1}\left|\frac{x}{a}\right|$

14.230 $\displaystyle\int (x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} - \frac{3a^2x\sqrt{x^2-a^2}}{8} + \frac{3}{8}a^4\ln(x+\sqrt{x^2-a^2})$

14.231 $\displaystyle\int x(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{5/2}}{5}$

14.232 $\displaystyle\int x^2(x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{5/2}}{6} + \frac{a^2x(x^2-a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2-a^2}}{16} + \frac{a^6}{16}\ln(x+\sqrt{x^2-a^2})$

14.233 $\displaystyle\int x^3(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{7/2}}{7} + \frac{a^2(x^2-a^2)^{5/2}}{5}$

14.234 $\displaystyle\int \frac{(x^2-a^2)^{3/2}}{x}\,dx = \frac{(x^2-a^2)^{3/2}}{3} - a^2\sqrt{x^2-a^2} + a^3\sec^{-1}\left|\frac{x}{a}\right|$

14.235 $\displaystyle\int \frac{(x^2-a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2-a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2-a^2}}{2} - \frac{3}{2}a^2\ln(x+\sqrt{x^2-a^2})$

14.236 $\displaystyle\int \frac{(x^2-a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2-a^2)^{3/2}}{2x^2} + \frac{3\sqrt{x^2-a^2}}{2} - \frac{3}{2}a\sec^{-1}\left|\frac{x}{a}\right|$

INTEGRALS INVOLVING $\sqrt{a^2-x^2}$

14.237 $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

14.238 $\displaystyle\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$

14.239 $\displaystyle\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

14.240 $\displaystyle\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = \frac{(a^2-x^2)^{3/2}}{3} - a^2\sqrt{a^2-x^2}$

14.241 $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

14.242 $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$

14.243 $\displaystyle\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

14.244 $$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

14.245 $$\int x\sqrt{a^2-x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{3}$$

14.246 $$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x(a^2-x^2)^{3/2}}{4} + \frac{a^2x\sqrt{a^2-x^2}}{8} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}$$

14.247 $$\int x^3\sqrt{a^2-x^2}\,dx = \frac{(a^2-x^2)^{5/2}}{5} - \frac{a^2(a^2-x^2)^{3/2}}{3}$$

14.248 $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.249 $$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$$

14.250 $$\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.251 $$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$$

14.252 $$\int \frac{x\,dx}{(a^2-x^2)^{3/2}} = \frac{1}{\sqrt{a^2-x^2}}$$

14.253 $$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}$$

14.254 $$\int \frac{x^3\,dx}{(a^2-x^2)^{3/2}} = \sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}$$

14.255 $$\int \frac{dx}{x(a^2-x^2)^{3/2}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.256 $$\int \frac{dx}{x^2(a^2-x^2)^{3/2}} = -\frac{\sqrt{a^2-x^2}}{a^4x} + \frac{x}{a^4\sqrt{a^2-x^2}}$$

14.257 $$\int \frac{dx}{x^3(a^2-x^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.258 $$\int (a^2-x^2)^{3/2}\,dx = \frac{x(a^2-x^2)^{3/2}}{4} + \frac{3a^2x\sqrt{a^2-x^2}}{8} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a}$$

14.259 $$\int x(a^2-x^2)^{3/2}\,dx = -\frac{(a^2-x^2)^{5/2}}{5}$$

14.260 $$\int x^2(a^2-x^2)^{3/2}\,dx = -\frac{x(a^2-x^2)^{5/2}}{6} + \frac{a^2x(a^2-x^2)^{3/2}}{24} + \frac{a^4x\sqrt{a^2-x^2}}{16} + \frac{a^6}{16}\sin^{-1}\frac{x}{a}$$

14.261 $$\int x^3(a^2-x^2)^{3/2}\,dx = \frac{(a^2-x^2)^{7/2}}{7} - \frac{a^2(a^2-x^2)^{5/2}}{5}$$

14.262 $$\int \frac{(a^2-x^2)^{3/2}}{x}\,dx = \frac{(a^2-x^2)^{3/2}}{3} + a^2\sqrt{a^2-x^2} - a^3\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.263 $$\int \frac{(a^2-x^2)^{3/2}}{x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{x} - \frac{3x\sqrt{a^2-x^2}}{2} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a}$$

14.264 $$\int \frac{(a^2-x^2)^{3/2}}{x^3}\,dx = -\frac{(a^2-x^2)^{3/2}}{2x^2} - \frac{3\sqrt{a^2-x^2}}{2} + \frac{3}{2}a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

INTEGRALS INVOLVING $ax^2 + bx + c$

14.265 $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}} \tan^{-1} \dfrac{2ax+b}{\sqrt{4ac-b^2}} \\[2ex] \dfrac{1}{\sqrt{b^2-4ac}} \ln\left(\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right) \end{cases}$$

If $b^2 = 4ac$, $ax^2 + bx + c = a(x + b/2a)^2$ and the results on pages 60-61 can be used. If $b = 0$ use results on page 64. If a or $c = 0$ use results on pages 60-61.

14.266 $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln(ax^2+bx+c) - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$$

14.267 $$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\ln(ax^2+bx+c) + \frac{b^2-2ac}{2a^2}\int \frac{dx}{ax^2+bx+c}$$

14.268 $$\int \frac{x^m\,dx}{ax^2+bx+c} = \frac{x^{m-1}}{(m-1)a} - \frac{c}{a}\int \frac{x^{m-2}\,dx}{ax^2+bx+c} - \frac{b}{a}\int \frac{x^{m-1}\,dx}{ax^2+bx+c}$$

14.269 $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\ln\left(\frac{x^2}{ax^2+bx+c}\right) - \frac{b}{2c}\int \frac{dx}{ax^2+bx+c}$$

14.270 $$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2}\ln\left(\frac{ax^2+bx+c}{x^2}\right) - \frac{1}{cx} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{ax^2+bx+c}$$

14.271 $$\int \frac{dx}{x^n(ax^2+bx+c)} = -\frac{1}{(n-1)cx^{n-1}} - \frac{b}{c}\int \frac{dx}{x^{n-1}(ax^2+bx+c)} - \frac{a}{c}\int \frac{dx}{x^{n-2}(ax^2+bx+c)}$$

14.272 $$\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

14.273 $$\int \frac{x\,dx}{(ax^2+bx+c)^2} = -\frac{bx+2c}{(4ac-b^2)(ax^2+bx+c)} - \frac{b}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

14.274 $$\int \frac{x^2\,dx}{(ax^2+bx+c)^2} = \frac{(b^2-2ac)x+bc}{a(4ac-b^2)(ax^2+bx+c)} + \frac{2c}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

14.275 $$\int \frac{x^m\,dx}{(ax^2+bx+c)^n} = -\frac{x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} + \frac{(m-1)c}{(2n-m-1)a}\int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^n}$$
$$- \frac{(n-m)b}{(2n-m-1)a}\int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^n}$$

14.276 $$\int \frac{x^{2n-1}\,dx}{(ax^2+bx+c)^n} = \frac{1}{a}\int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^{n-1}} - \frac{c}{a}\int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^n} - \frac{b}{a}\int \frac{x^{2n-2}\,dx}{(ax^2+bx+c)^n}$$

14.277 $$\int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^2} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)}$$

14.278 $$\int \frac{dx}{x^2(ax^2+bx+c)^2} = -\frac{1}{cx(ax^2+bx+c)} - \frac{3a}{c}\int \frac{dx}{(ax^2+bx+c)^2} - \frac{2b}{c}\int \frac{dx}{x(ax^2+bx+c)^2}$$

14.279 $$\int \frac{dx}{x^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^{n-1}} - \frac{(m+2n-3)a}{(m-1)c}\int \frac{dx}{x^{m-2}(ax^2+bx+c)^n}$$
$$- \frac{(m+n-2)b}{(m-1)c}\int \frac{dx}{x^{m-1}(ax^2+bx+c)^n}$$

INTEGRALS INVOLVING $\sqrt{ax^2+bx+c}$

In the following results if $b^2=4ac$, $\sqrt{ax^2+bx+c}=\sqrt{a}\,(x+b/2a)$ and the results on pages 60-61 can be used. If $b=0$ use the results on pages 67-70. If $a=0$ or $c=0$ use the results on pages 61-62.

14.280 $$\int\frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases}\dfrac{1}{\sqrt{a}}\ln\left(2\sqrt{a}\sqrt{ax^2+bx+c}+2ax+b\right)\\[2ex] -\dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{2ax+b}{\sqrt{b^2-4ac}}\right) \quad\text{or}\quad \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right)\end{cases}$$

14.281 $$\int\frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a}-\frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

14.282 $$\int\frac{x^2\,dx}{\sqrt{ax^2+bx+c}} = \frac{2ax-3b}{4a^2}\sqrt{ax^2+bx+c}+\frac{3b^2-4ac}{8a^2}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

14.283 $$\int\frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases}-\dfrac{1}{\sqrt{c}}\ln\left(\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right)\\[2ex] \dfrac{1}{\sqrt{-c}}\sin^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}\right) \quad\text{or}\quad -\dfrac{1}{\sqrt{c}}\sinh^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{4ac-b^2}}\right)\end{cases}$$

14.284 $$\int\frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{cx}-\frac{b}{2c}\int\frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.285 $$\int\sqrt{ax^2+bx+c}\,dx = \frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a}+\frac{4ac-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

14.286 $$\int x\sqrt{ax^2+bx+c}\,dx = \frac{(ax^2+bx+c)^{3/2}}{3a}-\frac{b(2ax+b)}{8a^2}\sqrt{ax^2+bx+c} - \frac{b(4ac-b^2)}{16a^2}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

14.287 $$\int x^2\sqrt{ax^2+bx+c}\,dx = \frac{6ax-5b}{24a^2}(ax^2+bx+c)^{3/2}+\frac{5b^2-4ac}{16a^2}\int\sqrt{ax^2+bx+c}\,dx$$

14.288 $$\int\frac{\sqrt{ax^2+bx+c}}{x}\,dx = \sqrt{ax^2+bx+c}+\frac{b}{2}\int\frac{dx}{\sqrt{ax^2+bx+c}}+c\int\frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.289 $$\int\frac{\sqrt{ax^2+bx+c}}{x^2}\,dx = -\frac{\sqrt{ax^2+bx+c}}{x}+a\int\frac{dx}{\sqrt{ax^2+bx+c}}+\frac{b}{2}\int\frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.290 $$\int\frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{2(2ax+b)}{(4ac-b^2)\sqrt{ax^2+bx+c}}$$

14.291 $$\int\frac{x\,dx}{(ax^2+bx+c)^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

14.292 $$\int\frac{x^2\,dx}{(ax^2+bx+c)^{3/2}} = \frac{(2b^2-4ac)x+2bc}{a(4ac-b^2)\sqrt{ax^2+bx+c}}+\frac{1}{a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

14.293 $$\int\frac{dx}{x(ax^2+bx+c)^{3/2}} = \frac{1}{c\sqrt{ax^2+bx+c}}+\frac{1}{c}\int\frac{dx}{x\sqrt{ax^2+bx+c}}-\frac{b}{2c}\int\frac{dx}{(ax^2+bx+c)^{3/2}}$$

14.294 $$\int\frac{dx}{x^2(ax^2+bx+c)^{3/2}} = -\frac{ax^2+2bx+c}{c^2x\sqrt{ax^2+bx+c}}+\frac{b^2-2ac}{2c^2}\int\frac{dx}{(ax^2+bx+c)^{3/2}} - \frac{3b}{2c^2}\int\frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.295 $$\int(ax^2+bx+c)^{n+1/2}\,dx = \frac{(2ax+b)(ax^2+bx+c)^{n+1/2}}{4a(n+1)} + \frac{(2n+1)(4ac-b^2)}{8a(n+1)}\int(ax^2+bx+c)^{n-1/2}\,dx$$

14.296 $$\int x(ax^2+bx+c)^{n+1/2}\,dx = \frac{(ax^2+bx+c)^{n+3/2}}{a(2n+3)} - \frac{b}{2a}\int (ax^2+bx+c)^{n+1/2}\,dx$$

14.297 $$\int \frac{dx}{(ax^2+bx+c)^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)(ax^2+bx+c)^{n-1/2}} + \frac{8a(n-1)}{(2n-1)(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^{n-1/2}}$$

14.298 $$\int \frac{dx}{x(ax^2+bx+c)^{n+1/2}} = \frac{1}{(2n-1)c(ax^2+bx+c)^{n-1/2}} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)^{n-1/2}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{n+1/2}}$$

INTEGRALS INVOLVING $x^3 + a^3$

Note that for formulas involving $x^3 - a^3$ replace a by $-a$.

14.299 $$\int \frac{dx}{x^3+a^3} = \frac{1}{6a^2}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{1}{a^2\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.300 $$\int \frac{x\,dx}{x^3+a^3} = \frac{1}{6a}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.301 $$\int \frac{x^2\,dx}{x^3+a^3} = \frac{1}{3}\ln(x^3+a^3)$$

14.302 $$\int \frac{dx}{x(x^3+a^3)} = \frac{1}{3a^3}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

14.303 $$\int \frac{dx}{x^2(x^3+a^3)} = -\frac{1}{a^3x} - \frac{1}{6a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} - \frac{1}{a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.304 $$\int \frac{dx}{(x^3+a^3)^2} = \frac{x}{3a^3(x^3+a^3)} + \frac{1}{9a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{2}{3a^5\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.305 $$\int \frac{x\,dx}{(x^3+a^3)^2} = \frac{x^2}{3a^3(x^3+a^3)} + \frac{1}{18a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{3a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.306 $$\int \frac{x^2\,dx}{(x^3+a^3)^2} = -\frac{1}{3(x^3+a^3)}$$

14.307 $$\int \frac{dx}{x(x^3+a^3)^2} = \frac{1}{3a^3(x^3+a^3)} + \frac{1}{3a^6}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

14.308 $$\int \frac{dx}{x^2(x^3+a^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(x^3+a^3)} - \frac{4}{3a^6}\int \frac{x\,dx}{x^3+a^3}$$ [See 14.300]

14.309 $$\int \frac{x^m\,dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3\int \frac{x^{m-3}\,dx}{x^3+a^3}$$

14.310 $$\int \frac{dx}{x^n(x^3+a^3)} = \frac{-1}{a^3(n-1)x^{n-1}} - \frac{1}{a^3}\int \frac{dx}{x^{n-3}(x^3+a^3)}$$

INTEGRALS INVOLVING $x^4 \pm a^4$

14.311 $$\int \frac{dx}{x^4+a^4} = \frac{1}{4a^3\sqrt{2}}\ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) - \frac{1}{2a^3\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

14.312 $$\int \frac{x\,dx}{x^4+a^4} = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}$$

14.313 $$\int \frac{x^2\,dx}{x^4+a^4} = \frac{1}{4a\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) - \frac{1}{2a\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

14.314 $$\int \frac{x^3\,dx}{x^4+a^4} = \frac{1}{4}\ln(x^4+a^4)$$

14.315 $$\int \frac{dx}{x(x^4+a^4)} = \frac{1}{4a^4}\ln\left(\frac{x^4}{x^4+a^4}\right)$$

14.316 $$\int \frac{dx}{x^2(x^4+a^4)} = -\frac{1}{a^4x} - \frac{1}{4a^5\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) + \frac{1}{2a^5\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

14.317 $$\int \frac{dx}{x^3(x^4+a^4)} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6}\tan^{-1}\frac{x^2}{a^2}$$

14.318 $$\int \frac{dx}{x^4-a^4} = \frac{1}{4a^3}\ln\left(\frac{x-a}{x+a}\right) - \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

14.319 $$\int \frac{x\,dx}{x^4-a^4} = \frac{1}{4a^2}\ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

14.320 $$\int \frac{x^2\,dx}{x^4-a^4} = \frac{1}{4a}\ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

14.321 $$\int \frac{x^3\,dx}{x^4-a^4} = \frac{1}{4}\ln(x^4-a^4)$$

14.322 $$\int \frac{dx}{x(x^4-a^4)} = \frac{1}{4a^4}\ln\left(\frac{x^4-a^4}{x^4}\right)$$

14.323 $$\int \frac{dx}{x^2(x^4-a^4)} = \frac{1}{a^4x} + \frac{1}{4a^5}\ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a^5}\tan^{-1}\frac{x}{a}$$

14.324 $$\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

INTEGRALS INVOLVING $x^n \pm a^n$

14.325 $$\int \frac{dx}{x(x^n+a^n)} = \frac{1}{na^n}\ln\frac{x^n}{x^n+a^n}$$

14.326 $$\int \frac{x^{n-1}\,dx}{x^n+a^n} = \frac{1}{n}\ln(x^n+a^n)$$

14.327 $$\int \frac{x^m\,dx}{(x^n+a^n)^r} = \int \frac{x^{m-n}\,dx}{(x^n+a^n)^{r-1}} - a^n\int\frac{x^{m-n}\,dx}{(x^n+a^n)^r}$$

14.328 $$\int \frac{dx}{x^m(x^n+a^n)^r} = \frac{1}{a^n}\int\frac{dx}{x^m(x^n+a^n)^{r-1}} - \frac{1}{a^n}\int\frac{dx}{x^{m-n}(x^n+a^n)^r}$$

14.329 $$\int \frac{dx}{x\sqrt{x^n+a^n}} = \frac{1}{n\sqrt{a^n}}\ln\left(\frac{\sqrt{x^n+a^n}-\sqrt{a^n}}{\sqrt{x^n+a^n}+\sqrt{a^n}}\right)$$

14.330 $$\int \frac{dx}{x(x^n-a^n)} = \frac{1}{na^n}\ln\left(\frac{x^n-a^n}{x^n}\right)$$

14.331 $$\int \frac{x^{n-1}\,dx}{x^n-a^n} = \frac{1}{n}\ln(x^n-a^n)$$

14.332 $$\int \frac{x^m\,dx}{(x^n-a^n)^r} = a^n\int\frac{x^{m-n}\,dx}{(x^n-a^n)^r} + \int\frac{x^{m-n}\,dx}{(x^n-a^n)^{r-1}}$$

14.333 $$\int \frac{dx}{x^m(x^n-a^n)^r} = \frac{1}{a^n}\int\frac{dx}{x^{m-n}(x^n-a^n)^r} - \frac{1}{a^n}\int\frac{dx}{x^m(x^n-a^n)^{r-1}}$$

14.334 $$\int \frac{dx}{x\sqrt{x^n-a^n}} = \frac{2}{n\sqrt{a^n}}\cos^{-1}\sqrt{\frac{a^n}{x^n}}$$

14.335 $$\int \frac{x^{p-1}\,dx}{x^{2m}+a^{2m}} = \frac{1}{ma^{2m-p}}\sum_{k=1}^{m} \sin\frac{(2k-1)p\pi}{2m}\tan^{-1}\left(\frac{x + a\cos[(2k-1)\pi/2m]}{a\sin[(2k-1)\pi/2m]}\right)$$
$$-\frac{1}{2ma^{2m-p}}\sum_{k=1}^{m}\cos\frac{(2k-1)p\pi}{2m}\ln\left(x^2 + 2ax\cos\frac{(2k-1)\pi}{2m} + a^2\right)$$
where $0 < p \leqq 2m$.

14.336 $$\int \frac{x^{p-1}\,dx}{x^{2m}-a^{2m}} = \frac{1}{2ma^{2m-p}}\sum_{k=1}^{m-1}\cos\frac{kp\pi}{m}\ln\left(x^2 - 2ax\cos\frac{k\pi}{m} + a^2\right)$$
$$-\frac{1}{ma^{2m-p}}\sum_{k=1}^{m-1}\sin\frac{kp\pi}{m}\tan^{-1}\left(\frac{x - a\cos(k\pi/m)}{a\sin(k\pi/m)}\right)$$
$$+\frac{1}{2ma^{2m-p}}\{\ln(x-a) + (-1)^p\ln(x+a)\}$$
where $0 < p \leqq 2m$.

14.337 $$\int \frac{x^{p-1}\,dx}{x^{2m+1}+a^{2m+1}}$$
$$= \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x + a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$
$$-\frac{(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2 + 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$$
$$+\frac{(-1)^{p-1}\ln(x+a)}{(2m+1)a^{2m-p+1}}$$
where $0 < p \leqq 2m+1$.

14.338 $$\int \frac{x^{p-1}\,dx}{x^{2m+1}-a^{2m+1}}$$
$$= \frac{-2}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x - a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$
$$+\frac{1}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2 - 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$$
$$+\frac{\ln(x-a)}{(2m+1)a^{2m-p+1}}$$
where $0 < p \leqq 2m+1$.

INTEGRALS INVOLVING $\sin ax$

14.339 $$\int \sin ax\,dx = -\frac{\cos ax}{a}$$

14.340 $$\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$$

14.341 $$\int x^2\sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$$

14.342 $$\int x^3\sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a^3} - \frac{x^3}{a}\right)\cos ax$$

14.343 $$\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \cdots$$

14.344 $$\int \frac{\sin ax}{x^2}\,dx = -\frac{\sin ax}{x} + a\int\frac{\cos ax}{x}\,dx$$ [see 14.373]

14.345 $$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\tan\frac{ax}{2}$$

14.346 $$\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.347 $$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

14.348 $\displaystyle\int x \sin^2 ax\, dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$

14.349 $\displaystyle\int \sin^3 ax\, dx = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$

14.350 $\displaystyle\int \sin^4 ax\, dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

14.351 $\displaystyle\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$

14.352 $\displaystyle\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a \sin^2 ax} + \frac{1}{2a}\ln \tan \frac{ax}{2}$

14.353 $\displaystyle\int \sin px \sin qx\, dx = \frac{\sin (p-q)x}{2(p-q)} - \frac{\sin (p+q)x}{2(p+q)}$ [If $p = \pm q$, see 14.368.]

14.354 $\displaystyle\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

14.355 $\displaystyle\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{2}{a^2}\ln \sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

14.356 $\displaystyle\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

14.357 $\displaystyle\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln \sin\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

14.358 $\displaystyle\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

14.359 $\displaystyle\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

14.360 $$\int \frac{dx}{p + q \sin ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}} \tan^{-1} \dfrac{p \tan \frac{1}{2}ax + q}{\sqrt{p^2-q^2}} \\[2ex] \dfrac{1}{a\sqrt{q^2-p^2}} \ln \left(\dfrac{p \tan \frac{1}{2}ax + q - \sqrt{q^2-p^2}}{p \tan \frac{1}{2}ax + q + \sqrt{q^2-p^2}}\right) \end{cases}$$

If $p = \pm q$ see 14.354 and 14.356.

14.361 $\displaystyle\int \frac{dx}{(p+q\sin ax)^2} = \frac{q \cos ax}{a(p^2-q^2)(p+q \sin ax)} + \frac{p}{p^2-q^2}\int \frac{dx}{p + q \sin ax}$

If $p = \pm q$ see 14.358 and 14.359.

14.362 $\displaystyle\int \frac{dx}{p^2 + q^2 \sin^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}} \tan^{-1} \frac{\sqrt{p^2+q^2}\tan ax}{p}$

14.363 $$\int \frac{dx}{p^2 - q^2 \sin^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}} \tan^{-1} \dfrac{\sqrt{p^2-q^2}\tan ax}{p} \\[2ex] \dfrac{1}{2ap\sqrt{q^2-p^2}} \ln \left(\dfrac{\sqrt{q^2-p^2}\tan ax + p}{\sqrt{q^2-p^2}\tan ax - p}\right) \end{cases}$$

14.364 $\displaystyle\int x^m \sin ax\, dx = -\frac{x^m \cos ax}{a} + \frac{mx^{m-1}\sin ax}{a^2} - \frac{m(m-1)}{a^2}\int x^{m-2}\sin ax\, dx$

14.365 $\displaystyle\int \frac{\sin ax}{x^n}\,dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\,dx$ [see 14.395]

14.366 $\displaystyle\int \sin^n ax\, dx = -\frac{\sin^{n-1} ax \cos ax}{an} + \frac{n-1}{n}\int \sin^{n-2} ax\, dx$

14.367 $\displaystyle\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{a(n-1)\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$

14.368 $\displaystyle\int \frac{x\,dx}{\sin^n ax} = \frac{-x \cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x\,dx}{\sin^{n-2} ax}$

INTEGRALS INVOLVING $\cos ax$

14.369 $$\int \cos ax\,dx = \frac{\sin ax}{a}$$

14.370 $$\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

14.371 $$\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

14.372 $$\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

14.373 $$\int \frac{\cos ax}{x}\,dx = \ln x - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$$

14.374 $$\int \frac{\cos ax}{x^2}\,dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax}{x}\,dx$$ [See 14.343]

14.375 $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.376 $$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.377 $$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

14.378 $$\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

14.379 $$\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a}$$

14.380 $$\int \cos^4 ax\,dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

14.381 $$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a}$$

14.382 $$\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.383 $$\int \cos ax\cos px\,dx = \frac{\sin(a-p)x}{2(a-p)} + \frac{\sin(a+p)x}{2(a+p)}$$ [If $a = \pm p$, see 14.377.]

14.384 $$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$$

14.385 $$\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\sin\frac{ax}{2}$$

14.386 $$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$$

14.387 $$\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\cos\frac{ax}{2}$$

14.388 $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

14.389 $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$$

14.390 $$\int \frac{dx}{p + q\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}} \tan^{-1} \sqrt{(p-q)/(p+q)} \tan \tfrac{1}{2}ax \\[2ex] \dfrac{1}{a\sqrt{q^2-p^2}} \ln\left(\dfrac{\tan\tfrac{1}{2}ax + \sqrt{(q+p)/(q-p)}}{\tan\tfrac{1}{2}ax - \sqrt{(q+p)/(q-p)}}\right) \end{cases}$$ [If $p = \pm q$ see 14.384 and 14.386.]

14.391 $$\int \frac{dx}{(p + q\cos ax)^2} = \frac{q\sin ax}{a(q^2-p^2)(p+q\cos ax)} - \frac{p}{q^2-p^2}\int\frac{dx}{p+q\cos ax}$$ [If $p = \pm q$ see 14.388 and 14.389.]

14.392 $$\int \frac{dx}{p^2 + q^2\cos^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}} \tan^{-1}\frac{p\tan ax}{\sqrt{p^2+q^2}}$$

14.393 $$\int \frac{dx}{p^2 - q^2\cos^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}} \tan^{-1}\dfrac{p\tan ax}{\sqrt{p^2-q^2}} \\[2ex] \dfrac{1}{2ap\sqrt{q^2-p^2}} \ln\left(\dfrac{p\tan ax - \sqrt{q^2-p^2}}{p\tan ax + \sqrt{q^2-p^2}}\right) \end{cases}$$

14.394 $$\int x^m \cos ax\, dx = \frac{x^m \sin ax}{a} + \frac{mx^{m-1}}{a^2}\cos ax - \frac{m(m-1)}{a^2}\int x^{m-2}\cos ax\, dx$$

14.395 $$\int \frac{\cos ax}{x^n}\, dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int\frac{\sin ax}{x^{n-1}}\, dx$$ [See 14.365]

14.396 $$\int \cos^n ax\, dx = \frac{\sin ax \cos^{n-1} ax}{an} + \frac{n-1}{n}\int \cos^{n-2} ax\, dx$$

14.397 $$\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} + \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2} ax}$$

14.398 $$\int \frac{x\, dx}{\cos^n ax} = \frac{x\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int\frac{x\, dx}{\cos^{n-2} ax}$$

INTEGRALS INVOLVING $\sin ax$ AND $\cos ax$

14.399 $$\int \sin ax \cos ax\, dx = \frac{\sin^2 ax}{2a}$$

14.400 $$\int \sin px \cos qx\, dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$$

14.401 $$\int \sin^n ax \cos ax\, dx = \frac{\sin^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.440.]

14.402 $$\int \cos^n ax \sin ax\, dx = -\frac{\cos^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.429.]

14.403 $$\int \sin^2 ax \cos^2 ax\, dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

14.404 $$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a}\ln\tan ax$$

14.405 $$\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a\sin ax}$$

14.406 $$\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a}\ln\tan\frac{ax}{2} + \frac{1}{a\cos ax}$$

14.407 $$\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2\cot 2ax}{a}$$

14.408 $$\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{\sin ax}{a} + \frac{1}{a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

14.409 $$\int \frac{\cos^2 ax}{\sin ax}\,dx = \frac{\cos ax}{a} + \frac{1}{a}\ln\tan\frac{ax}{2}$$

14.410 $$\int \frac{dx}{\cos ax(1\pm\sin ax)} = \mp\frac{1}{2a(1\pm\sin ax)} + \frac{1}{2a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

14.411 $$\int \frac{dx}{\sin ax(1\pm\cos ax)} = \pm\frac{1}{2a(1\pm\cos ax)} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

14.412 $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\ln\tan\left(\frac{ax}{2}\pm\frac{\pi}{8}\right)$$

14.413 $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

14.414 $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

14.415 $$\int \frac{\sin ax\,dx}{p + q\cos ax} = -\frac{1}{aq}\ln(p + q\cos ax)$$

14.416 $$\int \frac{\cos ax\,dx}{p + q\sin ax} = \frac{1}{aq}\ln(p + q\sin ax)$$

14.417 $$\int \frac{\sin ax\,dx}{(p + q\cos ax)^n} = \frac{1}{aq(n-1)(p+q\cos ax)^{n-1}}$$

14.418 $$\int \frac{\cos ax\,dx}{(p + q\sin ax)^n} = \frac{-1}{aq(n-1)(p+q\sin ax)^{n-1}}$$

14.419 $$\int \frac{dx}{p\sin ax + q\cos ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\tan\left(\frac{ax+\tan^{-1}(q/p)}{2}\right)$$

14.420 $$\int \frac{dx}{p\sin ax + q\cos ax + r} = \begin{cases} \dfrac{2}{a\sqrt{r^2-p^2-q^2}}\tan^{-1}\left(\dfrac{p+(r-q)\tan(ax/2)}{\sqrt{r^2-p^2-q^2}}\right) \\[2ex] \dfrac{1}{a\sqrt{p^2+q^2-r^2}}\ln\left(\dfrac{p-\sqrt{p^2+q^2-r^2}+(r-q)\tan(ax/2)}{p+\sqrt{p^2+q^2-r^2}+(r-q)\tan(ax/2)}\right) \end{cases}$$

If $r = q$ see 14.421. If $r^2 = p^2 + q^2$ see 14.422.

14.421 $$\int \frac{dx}{p\sin ax + q(1+\cos ax)} = \frac{1}{ap}\ln\left(q + p\tan\frac{ax}{2}\right)$$

14.422 $$\int \frac{dx}{p\sin ax + q\cos ax \pm\sqrt{p^2+q^2}} = \frac{-1}{a\sqrt{p^2+q^2}}\tan\left(\frac{\pi}{4}\mp\frac{ax+\tan^{-1}(q/p)}{2}\right)$$

14.423 $$\int \frac{dx}{p^2\sin^2 ax + q^2\cos^2 ax} = \frac{1}{apq}\tan^{-1}\left(\frac{p\tan ax}{q}\right)$$

14.424 $$\int \frac{dx}{p^2\sin^2 ax - q^2\cos^2 ax} = \frac{1}{2apq}\ln\left(\frac{p\tan ax - q}{p\tan ax + q}\right)$$

14.425 $$\int \sin^m ax\cos^n ax\,dx = \begin{cases} -\dfrac{\sin^{m-1}ax\cos^{n+1}ax}{a(m+n)} + \dfrac{m-1}{m+n}\displaystyle\int \sin^{m-2}ax\cos^n ax\,dx \\[2ex] \dfrac{\sin^{m+1}ax\cos^{n-1}ax}{a(m+n)} + \dfrac{n-1}{m+n}\displaystyle\int \sin^m ax\cos^{n-2}ax\,dx \end{cases}$$

14.426 $$\int \frac{\sin^m ax}{\cos^n ax}\,dx = \begin{cases} \dfrac{\sin^{m-1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n-2} ax}\,dx \\ \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^m ax}{\cos^{n-2} ax}\,dx \\ \dfrac{-\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^n ax}\,dx \end{cases}$$

14.427 $$\int \frac{\cos^m ax}{\sin^n ax}\,dx = \begin{cases} \dfrac{-\cos^{m-1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n-2} ax}\,dx \\ \dfrac{-\cos^{m+1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\cos^m ax}{\sin^{n-2} ax}\,dx \\ \dfrac{\cos^{m-1} ax}{a(m-n)\sin^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^n ax}\,dx \end{cases}$$

14.428 $$\int \frac{dx}{\sin^m ax \cos^n ax} = \begin{cases} \dfrac{1}{a(n-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax \cos^{n-2} ax} \\ \dfrac{-1}{a(m-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{\sin^{m-2} ax \cos^n ax} \end{cases}$$

INTEGRALS INVOLVING tan ax

14.429 $$\int \tan ax\,dx = -\frac{1}{a}\ln\cos ax = \frac{1}{a}\ln\sec ax$$

14.430 $$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$$

14.431 $$\int \tan^3 ax\,dx = \frac{\tan^2 ax}{2a} + \frac{1}{a}\ln\cos ax$$

14.432 $$\int \tan^n ax \sec^2 ax\,dx = \frac{\tan^{n+1} ax}{(n+1)a}$$

14.433 $$\int \frac{\sec^2 ax}{\tan ax}\,dx = \frac{1}{a}\ln\tan ax$$

14.434 $$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln\sin ax$$

14.435 $$\int x\tan ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} + \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.436 $$\int \frac{\tan ax}{x}\,dx = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.437 $$\int x\tan^2 ax\,dx = \frac{x\tan ax}{a} + \frac{1}{a^2}\ln\cos ax - \frac{x^2}{2}$$

14.438 $$\int \frac{dx}{p+q\tan ax} = \frac{px}{p^2+q^2} + \frac{q}{a(p^2+q^2)}\ln(q\sin ax + p\cos ax)$$

14.439 $$\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\,dx$$

INTEGRALS INVOLVING $\cot ax$

14.440 $\displaystyle\int \cot ax\,dx = \frac{1}{a}\ln\sin ax$

14.441 $\displaystyle\int \cot^2 ax\,dx = -\frac{\cot ax}{a} - x$

14.442 $\displaystyle\int \cot^3 ax\,dx = -\frac{\cot^2 ax}{2a} - \frac{1}{a}\ln\sin ax$

14.443 $\displaystyle\int \cot^n ax\csc^2 ax\,dx = -\frac{\cot^{n+1} ax}{(n+1)a}$

14.444 $\displaystyle\int \frac{\csc^2 ax}{\cot ax}\,dx = -\frac{1}{a}\ln\cot ax$

14.445 $\displaystyle\int \frac{dx}{\cot ax} = -\frac{1}{a}\ln\cos ax$

14.446 $\displaystyle\int x\cot ax\,dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{9} - \frac{(ax)^5}{225} - \cdots - \frac{2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} - \cdots\right\}$

14.447 $\displaystyle\int \frac{\cot ax}{x}\,dx = -\frac{1}{ax} - \frac{ax}{3} - \frac{(ax)^3}{135} - \cdots - \frac{2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} - \cdots$

14.448 $\displaystyle\int x\cot^2 ax\,dx = -\frac{x\cot ax}{a} + \frac{1}{a^2}\ln\sin ax - \frac{x^2}{2}$

14.449 $\displaystyle\int \frac{dx}{p+q\cot ax} = \frac{px}{p^2+q^2} - \frac{q}{a(p^2+q^2)}\ln(p\sin ax + q\cos ax)$

14.450 $\displaystyle\int \cot^n ax\,dx = -\frac{\cot^{n-1} ax}{(n-1)a} - \int \cot^{n-2} ax\,dx$

INTEGRALS INVOLVING $\sec ax$

14.451 $\displaystyle\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)$

14.452 $\displaystyle\int \sec^2 ax\,dx = \frac{\tan ax}{a}$

14.453 $\displaystyle\int \sec^3 ax\,dx = \frac{\sec ax\tan ax}{2a} + \frac{1}{2a}\ln(\sec ax + \tan ax)$

14.454 $\displaystyle\int \sec^n ax\tan ax\,dx = \frac{\sec^n ax}{na}$

14.455 $\displaystyle\int \frac{dx}{\sec ax} = \frac{\sin ax}{a}$

14.456 $\displaystyle\int x\sec ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

14.457 $\displaystyle\int \frac{\sec ax}{x}\,dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \cdots + \frac{E_n(ax)^{2n}}{2n(2n)!} + \cdots$

14.458 $\displaystyle\int x\sec^2 ax\,dx = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln\cos ax$

14.459 $$\int \frac{dx}{q + p \sec ax} = \frac{x}{q} - \frac{p}{q} \int \frac{dx}{p + q \cos ax}$$

14.460 $$\int \sec^n ax \, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax \, dx$$

INTEGRALS INVOLVING $\csc ax$

14.461 $$\int \csc ax \, dx = \frac{1}{a} \ln (\csc ax - \cot ax) = \frac{1}{a} \ln \tan \frac{ax}{2}$$

14.462 $$\int \csc^2 ax \, dx = -\frac{\cot ax}{a}$$

14.463 $$\int \csc^3 ax \, dx = -\frac{\csc ax \cot ax}{2a} + \frac{1}{2a} \ln \tan \frac{ax}{2}$$

14.464 $$\int \csc^n ax \cot ax \, dx = -\frac{\csc^n ax}{na}$$

14.465 $$\int \frac{dx}{\csc ax} = -\frac{\cos ax}{a}$$

14.466 $$\int x \csc ax \, dx = \frac{1}{a^2} \left\{ ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots \right\}$$

14.467 $$\int \frac{\csc ax}{x} dx = -\frac{1}{ax} + \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.468 $$\int x \csc^2 ax \, dx = -\frac{x \cot ax}{a} + \frac{1}{a^2} \ln \sin ax$$

14.469 $$\int \frac{dx}{q + p \csc ax} = \frac{x}{q} - \frac{p}{q} \int \frac{dx}{p + q \sin ax}$$ [See 14.360]

14.470 $$\int \csc^n ax \, dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1} \int \csc^{n-2} ax \, dx$$

INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

14.471 $$\int \sin^{-1} \frac{x}{a} dx = x \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2}$$

14.472 $$\int x \sin^{-1} \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{4}$$

14.473 $$\int x^2 \sin^{-1} \frac{x}{a} dx = \frac{x^3}{3} \sin^{-1} \frac{x}{a} + \frac{(x^2 + 2a^2)\sqrt{a^2 - x^2}}{9}$$

14.474 $$\int \frac{\sin^{-1}(x/a)}{x} dx = \frac{x}{a} + \frac{(x/a)^3}{2 \cdot 3 \cdot 3} + \frac{1 \cdot 3 (x/a)^5}{2 \cdot 4 \cdot 5 \cdot 5} + \frac{1 \cdot 3 \cdot 5 (x/a)^7}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7} + \cdots$$

14.475 $$\int \frac{\sin^{-1}(x/a)}{x^2} dx = -\frac{\sin^{-1}(x/a)}{x} - \frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right)$$

14.476 $$\int \left(\sin^{-1} \frac{x}{a} \right)^2 dx = x \left(\sin^{-1} \frac{x}{a} \right)^2 - 2x + 2\sqrt{a^2 - x^2} \sin^{-1} \frac{x}{a}$$

14.477 $$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$$

14.478 $$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{4}$$

14.479 $$\int x^2\cos^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cos^{-1}\frac{x}{a} - \frac{(x^2+2a^2)\sqrt{a^2-x^2}}{9}$$

14.480 $$\int \frac{\cos^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int\frac{\sin^{-1}(x/a)}{x}\,dx$$ [See 14.474]

14.481 $$\int \frac{\cos^{-1}(x/a)}{x^2}\,dx = -\frac{\cos^{-1}(x/a)}{x} + \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.482 $$\int\left(\cos^{-1}\frac{x}{a}\right)^2 dx = x\left(\cos^{-1}\frac{x}{a}\right)^2 - 2x - 2\sqrt{a^2-x^2}\cos^{-1}\frac{x}{a}$$

14.483 $$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$$

14.484 $$\int x\tan^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{ax}{2}$$

14.485 $$\int x^2\tan^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\tan^{-1}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(x^2+a^2)$$

14.486 $$\int \frac{\tan^{-1}(x/a)}{x}\,dx = \frac{x}{a} - \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} - \frac{(x/a)^7}{7^2} + \cdots$$

14.487 $$\int \frac{\tan^{-1}(x/a)}{x^2}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

14.488 $$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln(x^2+a^2)$$

14.489 $$\int x\cot^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a} + \frac{ax}{2}$$

14.490 $$\int x^2\cot^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cot^{-1}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(x^2+a^2)$$

14.491 $$\int \frac{\cot^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int\frac{\tan^{-1}(x/a)}{x}\,dx$$ [See 14.486]

14.492 $$\int \frac{\cot^{-1}(x/a)}{x^2}\,dx = -\frac{\cot^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

14.493 $$\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\dfrac{x}{a} - a\ln(x+\sqrt{x^2-a^2}) & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ x\sec^{-1}\dfrac{x}{a} + a\ln(x+\sqrt{x^2-a^2}) & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.494 $$\int x\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.495 $$\int x^2\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.496 $$\int \frac{\sec^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x + \frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots$$

14.497 $$\int \frac{\sec^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\sec^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ -\dfrac{\sec^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.498 $$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln(x+\sqrt{x^2-a^2}) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ x\csc^{-1}\dfrac{x}{a} - a\ln(x+\sqrt{x^2-a^2}) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.499 $$\int x\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.500 $$\int x^2\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.501 $$\int \frac{\csc^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots\right)$$

14.502 $$\int \frac{\csc^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\csc^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ -\dfrac{\csc^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.503 $$\int x^m\sin^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sin^{-1}\frac{x}{a} - \frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

14.504 $$\int x^m\cos^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cos^{-1}\frac{x}{a} + \frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

14.505 $$\int x^m\tan^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tan^{-1}\frac{x}{a} - \frac{a}{m+1}\int\frac{x^{m+1}}{x^2+a^2}\,dx$$

14.506 $$\int x^m\cot^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cot^{-1}\frac{x}{a} + \frac{a}{m+1}\int\frac{x^{m+1}}{x^2+a^2}\,dx$$

14.507 $$\int x^m\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{x^2-a^2}} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.508 $$\int x^m\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{x^2-a^2}} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

INTEGRALS INVOLVING e^{ax}

14.509 $\displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a}$

14.510 $\displaystyle\int xe^{ax}\,dx = \frac{e^{ax}}{a}\left(x-\frac{1}{a}\right)$

14.511 $\displaystyle\int x^2e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2-\frac{2x}{a}+\frac{2}{a^2}\right)$

14.512 $\displaystyle\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a}-\frac{n}{a}\int x^{n-1}e^{ax}\,dx$

$\displaystyle = \frac{e^{ax}}{a}\left(x^n-\frac{nx^{n-1}}{a}+\frac{n(n-1)x^{n-2}}{a^2}-\cdots\frac{(-1)^nn!}{a^n}\right)$ if n = positive integer

14.513 $\displaystyle\int\frac{e^{ax}}{x}\,dx = \ln x+\frac{ax}{1\cdot 1!}+\frac{(ax)^2}{2\cdot 2!}+\frac{(ax)^3}{3\cdot 3!}+\cdots$

14.514 $\displaystyle\int\frac{e^{ax}}{x^n}\,dx = \frac{-e^{ax}}{(n-1)x^{n-1}}+\frac{a}{n-1}\int\frac{e^{ax}}{x^{n-1}}\,dx$

14.515 $\displaystyle\int\frac{dx}{p+qe^{ax}} = \frac{x}{p}-\frac{1}{ap}\ln(p+qe^{ax})$

14.516 $\displaystyle\int\frac{dx}{(p+qe^{ax})^2} = \frac{x}{p^2}+\frac{1}{ap(p+qe^{ax})}-\frac{1}{ap^2}\ln(p+qe^{ax})$

14.517 $\displaystyle\int\frac{dx}{pe^{ax}+qe^{-ax}} = \begin{cases}\dfrac{1}{a\sqrt{pq}}\tan^{-1}\left(\sqrt{\dfrac{p}{q}}\,e^{ax}\right)\\[2ex] \dfrac{1}{2a\sqrt{-pq}}\ln\left(\dfrac{e^{ax}-\sqrt{-q/p}}{e^{ax}+\sqrt{-q/p}}\right)\end{cases}$

14.518 $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

14.519 $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$

14.520 $\displaystyle\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}-\frac{e^{ax}\{(a^2-b^2)\sin bx-2ab\cos bx\}}{(a^2+b^2)^2}$

14.521 $\displaystyle\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}-\frac{e^{ax}\{(a^2-b^2)\cos bx+2ab\sin bx\}}{(a^2+b^2)^2}$

14.522 $\displaystyle\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a}-\frac{1}{a}\int\frac{e^{ax}}{x}\,dx$

14.523 $\displaystyle\int e^{ax}\sin^n bx\,dx = \frac{e^{ax}\sin^{n-1}bx}{a^2+n^2b^2}(a\sin bx-nb\cos bx)+\frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2}bx\,dx$

14.524 $\displaystyle\int e^{ax}\cos^n bx\,dx = \frac{e^{ax}\cos^{n-1}bx}{a^2+n^2b^2}(a\cos bx+nb\sin bx)+\frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2}bx\,dx$

INTEGRALS INVOLVING $\ln x$

14.525 $\displaystyle\int \ln x\,dx \;=\; x\ln x - x$

14.526 $\displaystyle\int x\ln x\,dx \;=\; \frac{x^2}{2}(\ln x - \tfrac{1}{2})$

14.527 $\displaystyle\int x^m \ln x\,dx \;=\; \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right)$ [If $m=-1$ see 14.528.]

14.528 $\displaystyle\int \frac{\ln x}{x}\,dx \;=\; \frac{1}{2}\ln^2 x$

14.529 $\displaystyle\int \frac{\ln x}{x^2}\,dx \;=\; -\frac{\ln x}{x} - \frac{1}{x}$

14.530 $\displaystyle\int \ln^2 x\,dx \;=\; x\ln^2 x - 2x\ln x + 2x$

14.531 $\displaystyle\int \frac{\ln^n x\,dx}{x} \;=\; \frac{\ln^{n+1} x}{n+1}$ [If $n=-1$ see 14.532.]

14.532 $\displaystyle\int \frac{dx}{x\ln x} \;=\; \ln(\ln x)$

14.533 $\displaystyle\int \frac{dx}{\ln x} \;=\; \ln(\ln x) + \ln x + \frac{\ln^2 x}{2\cdot 2!} + \frac{\ln^3 x}{3\cdot 3!} + \cdots$

14.534 $\displaystyle\int \frac{x^m\,dx}{\ln x} \;=\; \ln(\ln x) + (m+1)\ln x + \frac{(m+1)^2\ln^2 x}{2\cdot 2!} + \frac{(m+1)^3\ln^3 x}{3\cdot 3!} + \cdots$

14.535 $\displaystyle\int \ln^n x\,dx \;=\; x\ln^n x - n\int \ln^{n-1} x\,dx$

14.536 $\displaystyle\int x^m \ln^n x\,dx \;=\; \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx$

If $m=-1$ see 14.531.

14.537 $\displaystyle\int \ln(x^2+a^2)\,dx \;=\; x\ln(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$

14.538 $\displaystyle\int \ln(x^2-a^2)\,dx \;=\; x\ln(x^2-a^2) - 2x + a\ln\left(\frac{x+a}{x-a}\right)$

14.539 $\displaystyle\int x^m \ln(x^2\pm a^2)\,dx \;=\; \frac{x^{m+1}\ln(x^2\pm a^2)}{m+1} - \frac{2}{m+1}\int \frac{x^{m+2}}{x^2\pm a^2}\,dx$

INTEGRALS INVOLVING $\sinh ax$

14.540 $\displaystyle\int \sinh ax\,dx \;=\; \frac{\cosh ax}{a}$

14.541 $\displaystyle\int x\sinh ax\,dx \;=\; \frac{x\cosh ax}{a} - \frac{\sinh ax}{a^2}$

14.542 $\displaystyle\int x^2\sinh ax\,dx \;=\; \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2}\sinh ax$

14.543 $\displaystyle\int \frac{\sinh ax}{x}\,dx = ax + \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} + \cdots$

14.544 $\displaystyle\int \frac{\sinh ax}{x^2}\,dx = -\frac{\sinh ax}{x} + a\int \frac{\cosh ax}{x}\,dx$ [See 14.565]

14.545 $\displaystyle\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln \tanh \frac{ax}{2}$

14.546 $\displaystyle\int \frac{x\,dx}{\sinh ax} = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} - \cdots + \frac{2(-1)^n(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$

14.547 $\displaystyle\int \sinh^2 ax\,dx = \frac{\sinh ax \cosh ax}{2a} - \frac{x}{2}$

14.548 $\displaystyle\int x\sinh^2 ax\,dx = \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2} - \frac{x^2}{4}$

14.549 $\displaystyle\int \frac{dx}{\sinh^2 ax} = -\frac{\coth ax}{a}$

14.550 $\displaystyle\int \sinh ax \sinh px\,dx = \frac{\sinh(a+p)x}{2(a+p)} - \frac{\sinh(a-p)x}{2(a-p)}$

For $a = \pm p$ see 14.547.

14.551 $\displaystyle\int \sinh ax \sin px\,dx = \frac{a\cosh ax \sin px - p\sinh ax \cos px}{a^2+p^2}$

14.552 $\displaystyle\int \sinh ax \cos px\,dx = \frac{a\cosh ax \cos px + p\sinh ax \sin px}{a^2+p^2}$

14.553 $\displaystyle\int \frac{dx}{p+q\sinh ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\left(\frac{qe^{ax}+p-\sqrt{p^2+q^2}}{qe^{ax}+p+\sqrt{p^2+q^2}}\right)$

14.554 $\displaystyle\int \frac{dx}{(p+q\sinh ax)^2} = \frac{-q\cosh ax}{a(p^2+q^2)(p+q\sinh ax)} + \frac{p}{p^2+q^2}\int \frac{dx}{p+q\sinh ax}$

14.555 $\displaystyle\int \frac{dx}{p^2+q^2\sinh^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{q^2-p^2}}\tan^{-1}\dfrac{\sqrt{q^2-p^2}\tanh ax}{p} \\[2ex] \dfrac{1}{2ap\sqrt{p^2-q^2}}\ln\left(\dfrac{p+\sqrt{p^2-q^2}\tanh ax}{p-\sqrt{p^2-q^2}\tanh ax}\right)\end{cases}$

14.556 $\displaystyle\int \frac{dx}{p^2-q^2\sinh^2 ax} = \frac{1}{2ap\sqrt{p^2+q^2}}\ln\left(\frac{p+\sqrt{p^2+q^2}\tanh ax}{p-\sqrt{p^2+q^2}\tanh ax}\right)$

14.557 $\displaystyle\int x^m \sinh ax\,dx = \frac{x^m\cosh ax}{a} - \frac{m}{a}\int x^{m-1}\cosh ax\,dx$ [See 14.585]

14.558 $\displaystyle\int \sinh^n ax\,dx = \frac{\sinh^{n-1} ax\cosh ax}{an} - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx$

14.559 $\displaystyle\int \frac{\sinh ax}{x^n}\,dx = \frac{-\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cosh ax}{x^{n-1}}\,dx$ [See 14.587]

14.560 $\displaystyle\int \frac{dx}{\sinh^n ax} = \frac{-\cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2} ax}$

14.561 $\displaystyle\int \frac{x\,dx}{\sinh^n ax} = \frac{-x\cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sinh^{n-2} ax} - \frac{n-2}{n-1}\int \frac{x\,dx}{\sinh^{n-2} ax}$

INTEGRALS INVOLVING $\cosh ax$

14.562 $$\int \cosh ax\,dx = \frac{\sinh ax}{a}$$

14.563 $$\int x \cosh ax\,dx = \frac{x \sinh ax}{a} - \frac{\cosh ax}{a^2}$$

14.564 $$\int x^2 \cosh ax\,dx = -\frac{2x \cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\sinh ax$$

14.565 $$\int \frac{\cosh ax}{x}\,dx = \ln x + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} + \frac{(ax)^6}{6\cdot 6!} + \cdots$$

14.566 $$\int \frac{\cosh ax}{x^2}\,dx = -\frac{\cosh ax}{x} + a\int \frac{\sinh ax}{x}\,dx \qquad \text{[See 14.543]}$$

14.567 $$\int \frac{dx}{\cosh ax} = \frac{2}{a}\tan^{-1} e^{ax}$$

14.568 $$\int \frac{x\,dx}{\cosh ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.569 $$\int \cosh^2 ax\,dx = \frac{x}{2} + \frac{\sinh ax \cosh ax}{2a}$$

14.570 $$\int x \cosh^2 ax\,dx = \frac{x^2}{4} + \frac{x \sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2}$$

14.571 $$\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a}$$

14.572 $$\int \cosh ax \cosh px\,dx = \frac{\sinh (a-p)x}{2(a-p)} + \frac{\sinh (a+p)x}{2(a+p)}$$

14.573 $$\int \cosh ax \sin px\,dx = \frac{a \sinh ax \sin px - p \cosh ax \cos px}{a^2 + p^2}$$

14.574 $$\int \cosh ax \cos px\,dx = \frac{a \sinh ax \cos px + p \cosh ax \sin px}{a^2 + p^2}$$

14.575 $$\int \frac{dx}{\cosh ax + 1} = \frac{1}{a}\tanh\frac{ax}{2}$$

14.576 $$\int \frac{dx}{\cosh ax - 1} = -\frac{1}{a}\coth\frac{ax}{2}$$

14.577 $$\int \frac{x\,dx}{\cosh ax + 1} = \frac{x}{a}\tanh\frac{ax}{2} - \frac{2}{a^2}\ln\cosh\frac{ax}{2}$$

14.578 $$\int \frac{x\,dx}{\cosh ax - 1} = -\frac{x}{a}\coth\frac{ax}{2} + \frac{2}{a^2}\ln\sinh\frac{ax}{2}$$

14.579 $$\int \frac{dx}{(\cosh ax + 1)^2} = \frac{1}{2a}\tanh\frac{ax}{2} - \frac{1}{6a}\tanh^3\frac{ax}{2}$$

14.580 $$\int \frac{dx}{(\cosh ax - 1)^2} = \frac{1}{2a}\coth\frac{ax}{2} - \frac{1}{6a}\coth^3\frac{ax}{2}$$

14.581 $$\int \frac{dx}{p + q\cosh ax} = \begin{cases} \dfrac{2}{a\sqrt{q^2-p^2}}\tan^{-1}\dfrac{qe^{ax}+p}{\sqrt{q^2-p^2}} \\[2ex] \dfrac{1}{a\sqrt{p^2-q^2}}\ln\left(\dfrac{qe^{ax}+p-\sqrt{p^2-q^2}}{qe^{ax}+p+\sqrt{p^2-q^2}}\right) \end{cases}$$

14.582 $$\int \frac{dx}{(p + q\cosh ax)^2} = \frac{q \sinh ax}{a(q^2-p^2)(p+q\cosh ax)} - \frac{p}{q^2-p^2}\int \frac{dx}{p+q\cosh ax}$$

14.583 $$\int \frac{dx}{p^2 - q^2\cosh^2 ax} = \begin{cases} \dfrac{1}{2ap\sqrt{p^2-q^2}} \ln\left(\dfrac{p\tanh ax + \sqrt{p^2-q^2}}{p\tanh ax - \sqrt{p^2-q^2}}\right) \\ \dfrac{-1}{ap\sqrt{q^2-p^2}} \tan^{-1}\dfrac{p\tanh ax}{\sqrt{q^2-p^2}} \end{cases}$$

14.584 $$\int \frac{dx}{p^2 + q^2\cosh^2 ax} = \begin{cases} \dfrac{1}{2ap\sqrt{p^2+q^2}} \ln\left(\dfrac{p\tanh ax + \sqrt{p^2+q^2}}{p\tanh ax - \sqrt{p^2+q^2}}\right) \\ \dfrac{1}{ap\sqrt{p^2+q^2}} \tan^{-1}\dfrac{p\tanh ax}{\sqrt{p^2+q^2}} \end{cases}$$

14.585 $$\int x^m \cosh ax\, dx = \frac{x^m \sinh ax}{a} - \frac{m}{a}\int x^{m-1}\sinh ax\, dx$$ [See 14.557]

14.586 $$\int \cosh^n ax\, dx = \frac{\cosh^{n-1} ax \sinh ax}{an} + \frac{n-1}{n}\int \cosh^{n-2} ax\, dx$$

14.587 $$\int \frac{\cosh ax}{x^n}\, dx = \frac{-\cosh ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\sinh ax}{x^{n-1}}\, dx$$ [See 14.559]

14.588 $$\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} ax}$$

14.589 $$\int \frac{x\, dx}{\cosh^n ax} = \frac{x\sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{1}{(n-1)(n-2)a^2\cosh^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x\, dx}{\cosh^{n-2} ax}$$

INTEGRALS INVOLVING $\sinh ax$ AND $\cosh ax$

14.590 $$\int \sinh ax \cosh ax\, dx = \frac{\sinh^2 ax}{2a}$$

14.591 $$\int \sinh px \cosh qx\, dx = \frac{\cosh (p+q)x}{2(p+q)} + \frac{\cosh (p-q)x}{2(p-q)}$$

14.592 $$\int \sinh^n ax \cosh ax\, dx = \frac{\sinh^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.615.]

14.593 $$\int \cosh^n ax \sinh ax\, dx = \frac{\cosh^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.604.]

14.594 $$\int \sinh^2 ax \cosh^2 ax\, dx = \frac{\sinh 4ax}{32a} - \frac{x}{8}$$

14.595 $$\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a}\ln\tanh ax$$

14.596 $$\int \frac{dx}{\sinh^2 ax \cosh ax} = -\frac{1}{a}\tan^{-1}\sinh ax - \frac{\operatorname{csch} ax}{a}$$

14.597 $$\int \frac{dx}{\sinh ax \cosh^2 ax} = \frac{\operatorname{sech} ax}{a} + \frac{1}{a}\ln\tanh\frac{ax}{2}$$

14.598 $$\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2\coth 2ax}{a}$$

14.599 $$\int \frac{\sinh^2 ax}{\cosh ax}\, dx = \frac{\sinh ax}{a} - \frac{1}{a}\tan^{-1}\sinh ax$$

14.600 $$\int \frac{\cosh^2 ax}{\sinh ax}\, dx = \frac{\cosh ax}{a} + \frac{1}{a}\ln\tanh\frac{ax}{2}$$

14.601 $$\int \frac{dx}{\cosh ax\,(1+\sinh ax)} = \frac{1}{2a}\ln\left(\frac{1+\sinh ax}{\cosh ax}\right) + \frac{1}{a}\tan^{-1} e^{ax}$$

14.602 $$\int \frac{dx}{\sinh ax\,(\cosh ax + 1)} = \frac{1}{2a}\ln\tanh\frac{ax}{2} + \frac{1}{2a(\cosh ax + 1)}$$

14.603 $$\int \frac{dx}{\sinh ax\,(\cosh ax - 1)} = -\frac{1}{2a}\ln\tanh\frac{ax}{2} - \frac{1}{2a(\cosh ax - 1)}$$

INTEGRALS INVOLVING $\tanh ax$

14.604 $$\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax$$

14.605 $$\int \tanh^2 ax\,dx = x - \frac{\tanh ax}{a}$$

14.606 $$\int \tanh^3 ax\,dx = \frac{1}{a}\ln\cosh ax - \frac{\tanh^2 ax}{2a}$$

14.607 $$\int \tanh^n ax\,\mathrm{sech}^2 ax\,dx = \frac{\tanh^{n+1} ax}{(n+1)a}$$

14.608 $$\int \frac{\mathrm{sech}^2 ax}{\tanh ax}\,dx = \frac{1}{a}\ln\tanh ax$$

14.609 $$\int \frac{dx}{\tanh ax} = \frac{1}{a}\ln\sinh ax$$

14.610 $$\int x\tanh ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} - \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.611 $$\int x\tanh^2 ax\,dx = \frac{x^2}{2} - \frac{x\tanh ax}{a} + \frac{1}{a^2}\ln\cosh ax$$

14.612 $$\int \frac{\tanh ax}{x}\,dx = ax - \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.613 $$\int \frac{dx}{p + q\tanh ax} = \frac{px}{p^2 - q^2} - \frac{q}{a(p^2 - q^2)}\ln(q\sinh ax + p\cosh ax)$$

14.614 $$\int \tanh^n ax\,dx = \frac{-\tanh^{n-1} ax}{a(n-1)} + \int \tanh^{n-2} ax\,dx$$

INTEGRALS INVOLVING $\coth ax$

14.615 $$\int \coth ax\,dx = \frac{1}{a}\ln\sinh ax$$

14.616 $$\int \coth^2 ax\,dx = x - \frac{\coth ax}{a}$$

14.617 $$\int \coth^3 ax\,dx = \frac{1}{a}\ln\sinh ax - \frac{\coth^2 ax}{2a}$$

14.618 $$\int \coth^n ax\,\mathrm{csch}^2 ax\,dx = -\frac{\coth^{n+1} ax}{(n+1)a}$$

14.619 $$\int \frac{\mathrm{csch}^2 ax}{\coth ax}\,dx = -\frac{1}{a}\ln\coth ax$$

14.620 $$\int \frac{dx}{\coth ax} = \frac{1}{a}\ln\cosh ax$$

14.621 $$\int x \coth ax\, dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{9} - \frac{(ax)^5}{225} + \cdots \frac{(-1)^{n-1}2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.622 $$\int x \coth^2 ax\, dx = \frac{x^2}{2} - \frac{x \coth ax}{a} + \frac{1}{a^2}\ln \sinh ax$$

14.623 $$\int \frac{\coth ax}{x}\, dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{(ax)^3}{135} + \cdots \frac{(-1)^n 2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.624 $$\int \frac{dx}{p + q \coth ax} = \frac{px}{p^2 - q^2} - \frac{q}{a(p^2 - q^2)} \ln (p \sinh ax + q \cosh ax)$$

14.625 $$\int \coth^n ax\, dx = -\frac{\coth^{n-1} ax}{a(n-1)} + \int \coth^{n-2} ax\, dx$$

INTEGRALS INVOLVING sech ax

14.626 $$\int \operatorname{sech} ax\, dx = \frac{2}{a} \tan^{-1} e^{ax}$$

14.627 $$\int \operatorname{sech}^2 ax\, dx = \frac{\tanh ax}{a}$$

14.628 $$\int \operatorname{sech}^3 ax\, dx = \frac{\operatorname{sech} ax \tanh ax}{2a} + \frac{1}{2a} \tan^{-1} \sinh ax$$

14.629 $$\int \operatorname{sech}^n ax \tanh ax\, dx = -\frac{\operatorname{sech}^n ax}{na}$$

14.630 $$\int \frac{dx}{\operatorname{sech} ax} = \frac{\sinh ax}{a}$$

14.631 $$\int x \operatorname{sech} ax\, dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.632 $$\int x \operatorname{sech}^2 ax\, dx = \frac{x \tanh ax}{a} - \frac{1}{a^2} \ln \cosh ax$$

14.633 $$\int \frac{\operatorname{sech} ax}{x}\, dx = \ln x - \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} - \frac{61(ax)^6}{4320} + \cdots \frac{(-1)^n E_n (ax)^{2n}}{2n(2n)!} + \cdots$$

14.634 $$\int \frac{dx}{q + p \operatorname{sech} ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q \cosh ax}$$ [See 14.581]

14.635 $$\int \operatorname{sech}^n ax\, dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{a(n-1)} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} ax\, dx$$

INTEGRALS INVOLVING csch ax

14.636 $$\int \operatorname{csch} ax\, dx = \frac{1}{a} \ln \tanh \frac{ax}{2}$$

14.637 $$\int \operatorname{csch}^2 ax\, dx = -\frac{\coth ax}{a}$$

14.638 $$\int \operatorname{csch}^3 ax\, dx = -\frac{\operatorname{csch} ax \coth ax}{2a} - \frac{1}{2a} \ln \tanh \frac{ax}{2}$$

14.639 $$\int \operatorname{csch}^n ax \coth ax\, dx = -\frac{\operatorname{csch}^n ax}{na}$$