14.621 $$\int x \coth ax\, dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{9} - \frac{(ax)^5}{225} + \cdots \frac{(-1)^{n-1}2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.622 $$\int x \coth^2 ax\, dx = \frac{x^2}{2} - \frac{x \coth ax}{a} + \frac{1}{a^2}\ln \sinh ax$$

14.623 $$\int \frac{\coth ax}{x}\, dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{(ax)^3}{135} + \cdots \frac{(-1)^n 2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.624 $$\int \frac{dx}{p + q \coth ax} = \frac{px}{p^2 - q^2} - \frac{q}{a(p^2 - q^2)} \ln (p \sinh ax + q \cosh ax)$$

14.625 $$\int \coth^n ax\, dx = -\frac{\coth^{n-1} ax}{a(n-1)} + \int \coth^{n-2} ax\, dx$$

INTEGRALS INVOLVING sech ax

14.626 $$\int \operatorname{sech} ax\, dx = \frac{2}{a}\tan^{-1} e^{ax}$$

14.627 $$\int \operatorname{sech}^2 ax\, dx = \frac{\tanh ax}{a}$$

14.628 $$\int \operatorname{sech}^3 ax\, dx = \frac{\operatorname{sech} ax \tanh ax}{2a} + \frac{1}{2a}\tan^{-1} \sinh ax$$

14.629 $$\int \operatorname{sech}^n ax \tanh ax\, dx = -\frac{\operatorname{sech}^n ax}{na}$$

14.630 $$\int \frac{dx}{\operatorname{sech} ax} = \frac{\sinh ax}{a}$$

14.631 $$\int x \operatorname{sech} ax\, dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots \frac{(-1)^n E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.632 $$\int x \operatorname{sech}^2 ax\, dx = \frac{x \tanh ax}{a} - \frac{1}{a^2}\ln \cosh ax$$

14.633 $$\int \frac{\operatorname{sech} ax}{x}\, dx = \ln x - \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} - \frac{61(ax)^6}{4320} + \cdots \frac{(-1)^n E_n(ax)^{2n}}{2n(2n)!} + \cdots$$

14.634 $$\int \frac{dx}{q + p \operatorname{sech} ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q \cosh ax}$$ [See 14.581]

14.635 $$\int \operatorname{sech}^n ax\, dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{a(n-1)} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} ax\, dx$$

INTEGRALS INVOLVING csch ax

14.636 $$\int \operatorname{csch} ax\, dx = \frac{1}{a}\ln \tanh \frac{ax}{2}$$

14.637 $$\int \operatorname{csch}^2 ax\, dx = -\frac{\coth ax}{a}$$

14.638 $$\int \operatorname{csch}^3 ax\, dx = -\frac{\operatorname{csch} ax \coth ax}{2a} - \frac{1}{2a}\ln \tanh \frac{ax}{2}$$

14.639 $$\int \operatorname{csch}^n ax \coth ax\, dx = -\frac{\operatorname{csch}^n ax}{na}$$

14.640 $$\int \frac{dx}{\operatorname{csch} ax} = \frac{1}{a} \cosh ax$$

14.641 $$\int x \operatorname{csch} ax \, dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(-1)^n(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.642 $$\int x \operatorname{csch}^2 ax \, dx = -\frac{x \coth ax}{a} + \frac{1}{a^2} \ln \sinh ax$$

14.643 $$\int \frac{\operatorname{csch} ax}{x} \, dx = -\frac{1}{ax} - \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots \frac{(-1)^n 2(2^{2n-1}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.644 $$\int \frac{dx}{q + p \operatorname{csch} ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q \sinh ax}$$ [See 14.563]

14.645 $$\int \operatorname{csch}^n ax \, dx = \frac{-\operatorname{csch}^{n-2} ax \coth ax}{a(n-1)} - \frac{n-2}{n-1}\int \operatorname{csch}^{n-2} ax \, dx$$

INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

14.646 $$\int \sinh^{-1}\frac{x}{a} \, dx = x \sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}$$

14.647 $$\int x \sinh^{-1}\frac{x}{a} \, dx = \left(\frac{x^2}{2} + \frac{a^2}{4}\right) \sinh^{-1}\frac{x}{a} - \frac{x\sqrt{x^2+a^2}}{4}$$

14.648 $$\int x^2 \sinh^{-1}\frac{x}{a} \, dx = \frac{x^3}{3} \sinh^{-1}\frac{x}{a} + \frac{(2a^2-x^2)\sqrt{x^2+a^2}}{9}$$

14.649 $$\int \frac{\sinh^{-1}(x/a)}{x} \, dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{2\cdot 3\cdot 3} + \dfrac{1\cdot 3(x/a)^5}{2\cdot 4\cdot 5\cdot 5} - \dfrac{1\cdot 3\cdot 5(x/a)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots & |x| < a \\ \dfrac{\ln^2(2x/a)}{2} - \dfrac{(a/x)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} - \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots & x > a \\ -\dfrac{\ln^2(-2x/a)}{2} + \dfrac{(a/x)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} - \cdots & x < -a \end{cases}$$

14.650 $$\int \frac{\sinh^{-1}(x/a)}{x^2} \, dx = -\frac{\sinh^{-1}(x/a)}{x} - \frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.651 $$\int \cosh^{-1}\frac{x}{a} \, dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

14.652 $$\int x \cosh^{-1}\frac{x}{a} \, dx = \begin{cases} \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) - \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) + \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

14.653 $$\int x^2 \cosh^{-1}\frac{x}{a} \, dx = \begin{cases} \frac{1}{3}x^3\cosh^{-1}(x/a) - \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \frac{1}{3}x^3\cosh^{-1}(x/a) + \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

14.654 $$\int \frac{\cosh^{-1}(x/a)}{x} \, dx = \pm\left[\frac{1}{2}\ln^2(2x/a) + \frac{(a/x)^2}{2\cdot 2\cdot 2} + \frac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots\right]$$

$+$ if $\cosh^{-1}(x/a) > 0$, $-$ if $\cosh^{-1}(x/a) < 0$

14.655 $$\int \frac{\cosh^{-1}(x/a)}{x^2} \, dx = -\frac{\cosh^{-1}(x/a)}{x} \mp \frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$ [$-$ if $\cosh^{-1}(x/a) > 0$, $+$ if $\cosh^{-1}(x/a) < 0$]

14.656 $$\int \tanh^{-1}\frac{x}{a} \, dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2)$$

14.657 $$\int x \tanh^{-1}\frac{x}{a} \, dx = \frac{ax}{2} + \frac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}$$

14.658 $$\int x^2 \tanh^{-1}\frac{x}{a} \, dx = \frac{ax^2}{6} + \frac{x^3}{3}\tanh^{-1}\frac{x}{a} + \frac{a^3}{6}\ln(a^2-x^2)$$

14.659 $$\int \frac{\tanh^{-1}(x/a)}{x}\,dx = \frac{x}{a} + \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} + \cdots$$

14.660 $$\int \frac{\tanh^{-1}(x/a)}{x^2}\,dx = -\frac{\tanh^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2}{a^2-x^2}\right)$$

14.661 $$\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}x + \frac{a}{2}\ln(x^2-a^2)$$

14.662 $$\int x\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \tfrac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}$$

14.663 $$\int x^2\coth^{-1}\frac{x}{a}\,dx = \frac{ax^2}{6} + \frac{x^3}{3}\coth^{-1}\frac{x}{a} + \frac{a^3}{6}\ln(x^2-a^2)$$

14.664 $$\int \frac{\coth^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{3^2} + \frac{(a/x)^5}{5^2} + \cdots\right)$$

14.665 $$\int \frac{\coth^{-1}(x/a)}{x^2}\,dx = -\frac{\coth^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2}{x^2-a^2}\right)$$

14.666 $$\int \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} x\operatorname{sech}^{-1}(x/a) + a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a) > 0 \\ x\operatorname{sech}^{-1}(x/a) - a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

14.667 $$\int x\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \tfrac{1}{2}x^2\operatorname{sech}^{-1}(x/a) - \tfrac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a) > 0 \\ \tfrac{1}{2}x^2\operatorname{sech}^{-1}(x/a) + \tfrac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

14.668 $$\int \frac{\operatorname{sech}^{-1}(x/a)}{x}\,dx = \begin{cases} -\tfrac{1}{2}\ln(a/x)\ln(4a/x) - \dfrac{(x/a)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots, & \operatorname{sech}^{-1}(x/a) > 0 \\ \tfrac{1}{2}\ln(a/x)\ln(4a/x) + \dfrac{(x/a)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

14.669 $$\int \operatorname{csch}^{-1}\frac{x}{a}\,dx = x\operatorname{csch}^{-1}\frac{x}{a} \pm a\sinh^{-1}\frac{x}{a} \qquad [+ \text{ if } x>0,\ - \text{ if } x<0]$$

14.670 $$\int x\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{x^2}{2}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a\sqrt{x^2+a^2}}{2} \qquad [+ \text{ if } x>0,\ - \text{ if } x<0]$$

14.671 $$\int \frac{\operatorname{csch}^{-1}(x/a)}{x}\,dx = \begin{cases} \tfrac{1}{2}\ln(x/a)\ln(4a/x) + \dfrac{1(x/a)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots & 0<x<a \\ \tfrac{1}{2}\ln(-x/a)\ln(-x/4a) - \dfrac{(x/a)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots & -a<x<0 \\ -\dfrac{a}{x} + \dfrac{(a/x)^3}{2\cdot 3\cdot 3} - \dfrac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \cdots & |x|>a \end{cases}$$

14.672 $$\int x^m\sinh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sinh^{-1}\frac{x}{a} - \frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{x^2+a^2}}\,dx$$

14.673 $$\int x^m\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}}{m+1}\cosh^{-1}\dfrac{x}{a} - \dfrac{1}{m+1}\displaystyle\int\frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) > 0 \\ \dfrac{x^{m+1}}{m+1}\cosh^{-1}\dfrac{x}{a} + \dfrac{1}{m+1}\displaystyle\int\frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) < 0 \end{cases}$$

14.674 $$\int x^m\tanh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tanh^{-1}\frac{x}{a} - \frac{a}{m+1}\int\frac{x^{m+1}}{a^2-x^2}\,dx$$

14.675 $$\int x^m\coth^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\coth^{-1}\frac{x}{a} - \frac{a}{m+1}\int\frac{x^{m+1}}{a^2-x^2}\,dx$$

14.676 $$\int x^m\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\dfrac{x}{a} + \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) > 0 \\ \dfrac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\dfrac{x}{a} - \dfrac{a}{m+1}\displaystyle\int\frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

14.677 $$\int x^m\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{m+1}\int\frac{x^m\,dx}{\sqrt{x^2+a^2}} \qquad [+ \text{ if } x>0,\ - \text{ if } x<0]$$

15 DEFINITE INTEGRALS

DEFINITION OF A DEFINITE INTEGRAL

Let $f(x)$ be defined in an interval $a \leqq x \leqq b$. Divide the interval into n equal parts of length $\Delta x = (b-a)/n$. Then the definite integral of $f(x)$ between $x = a$ and $x = b$ is defined as

15.1 $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \{f(a)\,\Delta x + f(a+\Delta x)\,\Delta x + f(a+2\Delta x)\,\Delta x + \cdots + f(a+(n-1)\,\Delta x)\,\Delta x\}$$

The limit will certainly exist if $f(x)$ is piecewise continuous.

If $f(x) = \frac{d}{dx}g(x)$, then by the fundamental theorem of the integral calculus the above definite integral can be evaluated by using the result

15.2 $$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}g(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)$$

If the interval is infinite or if $f(x)$ has a singularity at some point in the interval, the definite integral is called an *improper integral* and can be defined by using appropriate limiting procedures. For example,

15.3 $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

15.4 $$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(x)\,dx$$

15.5 $$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{b-\epsilon} f(x)\,dx \qquad \text{if } b \text{ is a singular point}$$

15.6 $$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^{b} f(x)\,dx \qquad \text{if } a \text{ is a singular point}$$

GENERAL FORMULAS INVOLVING DEFINITE INTEGRALS

15.7 $$\int_a^b \{f(x) \pm g(x) \pm h(x) \pm \cdots\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \pm \int_a^b h(x)\,dx \pm \cdots$$

15.8 $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \qquad \text{where } c \text{ is any constant}$$

15.9 $$\int_a^a f(x)\,dx = 0$$

15.10 $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

15.11 $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

15.12 $$\int_a^b f(x)\,dx = (b-a)\,f(c) \qquad \text{where } c \text{ is between } a \text{ and } b$$

This is called the *mean value theorem* for definite integrals and is valid if $f(x)$ is continuous in $a \leqq x \leqq b$.

15.13 $$\int_a^b f(x)\,g(x)\,dx \;=\; f(c)\int_a^b g(x)\,dx \qquad \text{where } c \text{ is between } a \text{ and } b$$

This is a generalization of 15.12 and is valid if $f(x)$ and $g(x)$ are continuous in $a \leqq x \leqq b$ and $g(x) \geqq 0$.

LEIBNITZ'S RULE FOR DIFFERENTIATION OF INTEGRALS

15.14 $$\frac{d}{d\alpha}\int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)\,dx \;=\; \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}\,dx + F(\phi_2,\alpha)\frac{d\phi_2}{d\alpha} - F(\phi_1,\alpha)\frac{d\phi_1}{d\alpha}$$

APPROXIMATE FORMULAS FOR DEFINITE INTEGRALS

In the following the interval from $x = a$ to $x = b$ is subdivided into n equal parts by the points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ and we let $y_0 = f(x_0),\ y_1 = f(x_1),\ y_2 = f(x_2),\ \ldots,\ y_n = f(x_n),\ h = (b-a)/n$.

Rectangular formula

15.15 $$\int_a^b f(x)\,dx \;\approx\; h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$$

Trapezoidal formula

15.16 $$\int_a^b f(x)\,dx \;\approx\; \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

Simpson's formula (or parabolic formula) for n even

15.17 $$\int_a^b f(x)\,dx \;\approx\; \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

DEFINITE INTEGRALS INVOLVING RATIONAL OR IRRATIONAL EXPRESSIONS

15.18 $$\int_0^\infty \frac{dx}{x^2 + a^2} \;=\; \frac{\pi}{2a}$$

15.19 $$\int_0^\infty \frac{x^{p-1}\,dx}{1+x} \;=\; \frac{\pi}{\sin p\pi}, \qquad 0 < p < 1$$

15.20 $$\int_0^\infty \frac{x^m\,dx}{x^n + a^n} \;=\; \frac{\pi a^{m+1-n}}{n \sin[(m+1)\pi/n]}, \qquad 0 < m+1 < n$$

15.21 $$\int_0^\infty \frac{x^m\,dx}{1 + 2x\cos\beta + x^2} \;=\; \frac{\pi}{\sin m\pi}\,\frac{\sin m\beta}{\sin \beta}$$

15.22 $$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \;=\; \frac{\pi}{2}$$

15.23 $$\int_0^a \sqrt{a^2 - x^2}\,dx \;=\; \frac{\pi a^2}{4}$$

15.24 $$\int_0^a x^m (a^n - x^n)^p\,dx \;=\; \frac{a^{m+1+np}\,\Gamma[(m+1)/n]\,\Gamma(p+1)}{n\Gamma[(m+1)/n + p + 1]}$$

15.25 $$\int_0^\infty \frac{x^m\,dx}{(x^n + a^n)^r} \;=\; \frac{(-1)^{r-1}\pi a^{m+1-nr}\,\Gamma[(m+1)/n]}{n \sin[(m+1)\pi/n]\,(r-1)!\,\Gamma[(m+1)/n - r + 1]}, \qquad 0 < m+1 < nr$$

DEFINITE INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

All letters are considered positive unless otherwise indicated.

15.26 $$\int_0^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m, n \text{ integers and } m \neq n \\ \pi/2 & m, n \text{ integers and } m = n \end{cases}$$

15.27 $$\int_0^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m, n \text{ integers and } m \neq n \\ \pi/2 & m, n \text{ integers and } m = n \end{cases}$$

15.28 $$\int_0^{\pi} \sin mx \cos nx \, dx = \begin{cases} 0 & m, n \text{ integers and } m + n \text{ even} \\ 2m/(m^2 - n^2) & m, n \text{ integers and } m + n \text{ odd} \end{cases}$$

15.29 $$\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4}$$

15.30 $$\int_0^{\pi/2} \sin^{2m} x \, dx = \int_0^{\pi/2} \cos^{2m} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots 2m-1}{2 \cdot 4 \cdot 6 \cdots 2m} \frac{\pi}{2}, \qquad m = 1, 2, \ldots$$

15.31 $$\int_0^{\pi/2} \sin^{2m+1} x \, dx = \int_0^{\pi/2} \cos^{2m+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots 2m}{1 \cdot 3 \cdot 5 \cdots 2m+1}, \qquad m = 1, 2, \ldots$$

15.32 $$\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x \, dx = \frac{\Gamma(p)\,\Gamma(q)}{2\,\Gamma(p+q)}$$

15.33 $$\int_0^{\infty} \frac{\sin px}{x} \, dx = \begin{cases} \pi/2 & p > 0 \\ 0 & p = 0 \\ -\pi/2 & p < 0 \end{cases}$$

15.34 $$\int_0^{\infty} \frac{\sin px \cos qx}{x} \, dx = \begin{cases} 0 & p > q > 0 \\ \pi/2 & 0 < p < q \\ \pi/4 & p = q > 0 \end{cases}$$

15.35 $$\int_0^{\infty} \frac{\sin px \sin qx}{x^2} \, dx = \begin{cases} \pi p/2 & 0 < p \leqq q \\ \pi q/2 & p \geqq q > 0 \end{cases}$$

15.36 $$\int_0^{\infty} \frac{\sin^2 px}{x^2} \, dx = \frac{\pi p}{2}$$

15.37 $$\int_0^{\infty} \frac{1 - \cos px}{x^2} \, dx = \frac{\pi p}{2}$$

15.38 $$\int_0^{\infty} \frac{\cos px - \cos qx}{x} \, dx = \ln \frac{q}{p}$$

15.39 $$\int_0^{\infty} \frac{\cos px - \cos qx}{x^2} \, dx = \frac{\pi(q-p)}{2}$$

15.40 $$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2} \, dx = \frac{\pi}{2a} e^{-ma}$$

15.41 $$\int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} \, dx = \frac{\pi}{2} e^{-ma}$$

15.42 $$\int_0^{\infty} \frac{\sin mx}{x(x^2 + a^2)} \, dx = \frac{\pi}{2a^2}(1 - e^{-ma})$$

15.43 $$\int_0^{2\pi} \frac{dx}{a + b \sin x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

15.44 $$\int_0^{2\pi} \frac{dx}{a + b \cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

15.45 $$\int_0^{\pi/2} \frac{dx}{a + b \cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2 - b^2}}$$

15.46 $$\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \int_0^{2\pi} \frac{dx}{(a + b\cos x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

15.47 $$\int_0^{2\pi} \frac{dx}{1 - 2a\cos x + a^2} = \frac{2\pi}{1 - a^2}, \quad 0 < a < 1$$

15.48 $$\int_0^{\pi} \frac{x\sin x\,dx}{1 - 2a\cos x + a^2} = \begin{cases} (\pi/a)\ln(1+a) & |a| < 1 \\ \pi\ln(1 + 1/a) & |a| > 1 \end{cases}$$

15.49 $$\int_0^{\pi} \frac{\cos mx\,dx}{1 - 2a\cos x + a^2} = \frac{\pi a^m}{1 - a^2}, \quad a^2 < 1, \quad m = 0, 1, 2, \ldots$$

15.50 $$\int_0^{\infty} \sin ax^2\,dx = \int_0^{\infty} \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$

15.51 $$\int_0^{\infty} \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \quad n > 1$$

15.52 $$\int_0^{\infty} \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \quad n > 1$$

15.53 $$\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\,dx = \int_0^{\infty} \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

15.54 $$\int_0^{\infty} \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \quad 0 < p < 1$$

15.55 $$\int_0^{\infty} \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \quad 0 < p < 1$$

15.56 $$\int_0^{\infty} \sin ax^2 \cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right)$$

15.57 $$\int_0^{\infty} \cos ax^2 \cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$$

15.58 $$\int_0^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

15.59 $$\int_0^{\infty} \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

15.60 $$\int_0^{\infty} \frac{\tan x}{x}\,dx = \frac{\pi}{2}$$

15.61 $$\int_0^{\pi/2} \frac{dx}{1 + \tan^m x} = \frac{\pi}{4}$$

15.62 $$\int_0^{\pi/2} \frac{x}{\sin x}\,dx = 2\left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right\}$$

15.63 $$\int_0^{1} \frac{\tan^{-1} x}{x}\,dx = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

15.64 $$\int_0^{1} \frac{\sin^{-1} x}{x}\,dx = \frac{\pi}{2}\ln 2$$

15.65 $$\int_0^{1} \frac{1 - \cos x}{x}\,dx - \int_1^{\infty} \frac{\cos x}{x}\,dx = \gamma$$

15.66 $$\int_0^{\infty} \left(\frac{1}{1 + x^2} - \cos x\right)\frac{dx}{x} = \gamma$$

15.67 $$\int_0^{\infty} \frac{\tan^{-1} px - \tan^{-1} qx}{x}\,dx = \frac{\pi}{2}\ln\frac{p}{q}$$

DEFINITE INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

15.68 $$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

15.69 $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

15.70 $$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$$

15.71 $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$$

15.72 $$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

15.73 $$\int_0^\infty e^{-ax^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-b^2/4a}$$

15.74 $$\int_0^\infty e^{-(ax^2+bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}\operatorname{erfc}\frac{b}{2\sqrt{a}}$$

where $$\operatorname{erfc}(p) = \frac{2}{\sqrt{\pi}}\int_p^\infty e^{-x^2}\,dx$$

15.75 $$\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}$$

15.76 $$\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

15.77 $$\int_0^\infty x^m e^{-ax^2}\,dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$$

15.78 $$\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}$$

15.79 $$\int_0^\infty \frac{x\,dx}{e^x-1} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{\pi^2}{6}$$

15.80 $$\int_0^\infty \frac{x^{n-1}}{e^x-1}\,dx = \Gamma(n)\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right)$$

For even n this can be summed in terms of Bernoulli numbers [see pages 108-109 and 114-115].

15.81 $$\int_0^\infty \frac{x\,dx}{e^x+1} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}$$

15.82 $$\int_0^\infty \frac{x^{n-1}}{e^x+1}\,dx = \Gamma(n)\left(\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\cdots\right)$$

For some positive integer values of n the series can be summed [see pages 108-109 and 114-115].

15.83 $$\int_0^\infty \frac{\sin mx}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{m}{2}-\frac{1}{2m}$$

15.84 $$\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x} = \gamma$$

15.85 $$\int_0^\infty \frac{e^{-x^2}-e^{-x}}{x}\,dx = \tfrac{1}{2}\gamma$$

15.86 $$\int_0^\infty \left(\frac{1}{e^x-1}-\frac{e^{-x}}{x}\right)dx = \gamma$$

15.87 $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x \sec px}\,dx = \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right)$$

15.88 $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x \csc px}\,dx = \tan^{-1}\frac{b}{p} - \tan^{-1}\frac{a}{p}$$

15.89 $$\int_0^\infty \frac{e^{-ax}(1-\cos x)}{x^2}\,dx = \cot^{-1} a - \frac{a}{2}\ln(a^2+1)$$

DEFINITE INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

15.90 $$\int_0^1 x^m(\ln x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}} \qquad m>-1,\ n=0,1,2,\ldots$$

If $n \neq 0,1,2,\ldots$ replace $n!$ by $\Gamma(n+1)$.

15.91 $$\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$$

15.92 $$\int_0^1 \frac{\ln x}{1-x}\,dx = -\frac{\pi^2}{6}$$

15.93 $$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$

15.94 $$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$

15.95 $$\int_0^1 \ln x \ln(1+x)\,dx = 2 - 2\ln 2 - \frac{\pi^2}{12}$$

15.96 $$\int_0^1 \ln x \ln(1-x)\,dx = 2 - \frac{\pi^2}{6}$$

15.97 $$\int_0^\infty \frac{x^{p-1}\ln x}{1+x}\,dx = -\pi^2 \csc p\pi \cot p\pi \qquad 0<p<1$$

15.98 $$\int_0^1 \frac{x^m - x^n}{\ln x}\,dx = \ln\frac{m+1}{n+1}$$

15.99 $$\int_0^\infty e^{-x}\ln x\,dx = -\gamma$$

15.100 $$\int_0^\infty e^{-x^2}\ln x\,dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln 2)$$

15.101 $$\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$$

15.102 $$\int_0^{\pi/2} \ln\sin x\,dx = \int_0^{\pi/2}\ln\cos x\,dx = -\frac{\pi}{2}\ln 2$$

15.103 $$\int_0^{\pi/2} (\ln\sin x)^2\,dx = \int_0^{\pi/2}(\ln\cos x)^2\,dx = \frac{\pi}{2}(\ln 2)^2 + \frac{\pi^3}{24}$$

15.104 $$\int_0^\pi x\ln\sin x\,dx = -\frac{\pi^2}{2}\ln 2$$

15.105 $$\int_0^{\pi/2} \sin x\ln\sin x\,dx = \ln 2 - 1$$

15.106 $$\int_0^{2\pi} \ln(a+b\sin x)\,dx = \int_0^{2\pi}\ln(a+b\cos x)\,dx = 2\pi\ln(a+\sqrt{a^2-b^2})$$

15.107 $$\int_0^{\pi} \ln(a + b\cos x)\,dx = \pi \ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$$

15.108 $$\int_0^{\pi} \ln(a^2 - 2ab\cos x + b^2)\,dx = \begin{cases} 2\pi \ln a, & a \geqq b > 0 \\ 2\pi \ln b, & b \geqq a > 0 \end{cases}$$

15.109 $$\int_0^{\pi/4} \ln(1 + \tan x)\,dx = \frac{\pi}{8}\ln 2$$

15.110 $$\int_0^{\pi/2} \sec x \ln\left(\frac{1+b\cos x}{1+a\cos x}\right) dx = \tfrac{1}{2}\{(\cos^{-1} a)^2 - (\cos^{-1} b)^2\}$$

15.111 $$\int_0^{a} \ln\left(2\sin\frac{x}{2}\right) dx = -\left(\frac{\sin a}{1^2} + \frac{\sin 2a}{2^2} + \frac{\sin 3a}{3^2} + \cdots\right)$$

See also 15.102.

DEFINITE INTEGRALS INVOLVING HYPERBOLIC FUNCTIONS

15.112 $$\int_0^{\infty} \frac{\sin ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b}$$

15.113 $$\int_0^{\infty} \frac{\cos ax}{\cosh bx}\,dx = \frac{\pi}{2b}\operatorname{sech}\frac{a\pi}{2b}$$

15.114 $$\int_0^{\infty} \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

15.115 $$\int_0^{\infty} \frac{x^n\,dx}{\sinh ax} = \frac{2^{n+1}-1}{2^n a^{n+1}}\Gamma(n+1)\left\{\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \cdots\right\}$$

If n is an odd positive integer, the series can be summed [see page 108].

15.116 $$\int_0^{\infty} \frac{\sinh ax}{e^{bx}+1}\,dx = \frac{\pi}{2b}\csc\frac{a\pi}{b} - \frac{1}{2a}$$

15.117 $$\int_0^{\infty} \frac{\sinh ax}{e^{bx}-1}\,dx = \frac{1}{2a} - \frac{\pi}{2b}\cot\frac{a\pi}{b}$$

MISCELLANEOUS DEFINITE INTEGRALS

15.118 $$\int_0^{\infty} \frac{f(ax) - f(bx)}{x}\,dx = \{f(0) - f(\infty)\}\ln\frac{b}{a}$$

This is called *Frullani's integral.* It holds if $f'(x)$ is continuous and $\int_1^{\infty} \frac{f(x)-f(\infty)}{x}\,dx$ converges.

15.119 $$\int_0^1 \frac{dx}{x^x} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots$$

15.120 $$\int_{-a}^{a} (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

16 THE GAMMA FUNCTION

DEFINITION OF THE GAMMA FUNCTION $\Gamma(n)$ FOR $n > 0$

16.1 $$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \qquad n > 0$$

RECURSION FORMULA

16.2 $$\Gamma(n+1) = n\,\Gamma(n)$$

16.3 $$\Gamma(n+1) = n! \quad \text{if } n = 0, 1, 2, \ldots \text{ where } 0! = 1$$

THE GAMMA FUNCTION FOR $n < 0$

For $n < 0$ the gamma function can be defined by using 16.2, i.e.

16.4 $$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

GRAPH OF THE GAMMA FUNCTION

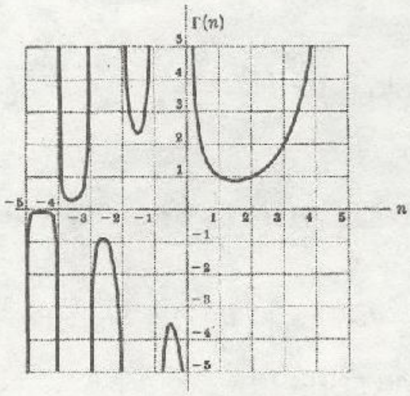


Fig. 16-1

SPECIAL VALUES FOR THE GAMMA FUNCTION

16.5 $$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

16.6 $$\Gamma(m + \tfrac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m}\sqrt{\pi} \qquad m = 1, 2, 3, \ldots$$

16.7 $$\Gamma(-m + \tfrac{1}{2}) = \frac{(-1)^m 2^m \sqrt{\pi}}{1 \cdot 3 \cdot 5 \cdots (2m-1)} \qquad m = 1, 2, 3, \ldots$$

RELATIONSHIPS AMONG GAMMA FUNCTIONS

16.8 $$\Gamma(p)\,\Gamma(1-p) \;=\; \frac{\pi}{\sin p\pi}$$

16.9 $$2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) \;=\; \sqrt{\pi}\,\Gamma(2x)$$

This is called the *duplication formula.*

16.10 $$\Gamma(x)\,\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(x+\frac{m-1}{m}\right) \;=\; m^{\frac{1}{2}-mx}\,(2\pi)^{(m-1)/2}\,\Gamma(mx)$$

For $m=2$ this reduces to 16.9.

OTHER DEFINITIONS OF THE GAMMA FUNCTION

16.11 $$\Gamma(x+1) \;=\; \lim_{k\to\infty} \frac{1\cdot 2\cdot 3\cdots k}{(x+1)(x+2)\cdots(x+k)}\,k^x$$

16.12 $$\frac{1}{\Gamma(x)} \;=\; xe^{\gamma x}\prod_{m=1}^{\infty}\left\{\left(1+\frac{x}{m}\right)e^{-x/m}\right\}$$

This is an infinite product representation for the gamma function where γ is Euler's constant.

DERIVATIVES OF THE GAMMA FUNCTION

16.13 $$\Gamma'(1) \;=\; \int_0^\infty e^{-x}\ln x\,dx \;=\; -\gamma$$

16.14 $$\frac{\Gamma'(x)}{\Gamma(x)} \;=\; -\gamma+\left(\frac{1}{1}-\frac{1}{x}\right)+\left(\frac{1}{2}-\frac{1}{x+1}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{x+n-1}\right)+\cdots$$

ASYMPTOTIC EXPANSIONS FOR THE GAMMA FUNCTION

16.15 $$\Gamma(x+1) \;=\; \sqrt{2\pi x}\,x^x e^{-x}\left\{1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51{,}840x^3}+\cdots\right\}$$

This is called *Stirling's asymptotic series.*

If we let $x=n$ a positive integer in 16.15, then a useful approximation for $n!$ where n is large [e.g. $n>10$] is given by *Stirling's formula*

16.16 $$n! \;\sim\; \sqrt{2\pi n}\,n^n e^{-n}$$

where $\sim$ is used to indicate that the ratio of the terms on each side approaches 1 as $n\to\infty$.

MISCELLANEOUS RESULTS

16.17 $$|\Gamma(ix)|^2 \;=\; \frac{\pi}{x\sinh \pi x}$$

17 THE BETA FUNCTION

DEFINITION OF THE BETA FUNCTION $B(m,n)$

17.1 $$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \qquad m>0,\ n>0$$

RELATIONSHIP OF BETA FUNCTION TO GAMMA FUNCTION

17.2 $$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Extensions of $B(m,n)$ to $m<0$, $n<0$ is provided by using 16.4, page 101.

SOME IMPORTANT RESULTS

17.3 $$B(m,n) = B(n,m)$$

17.4 $$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

17.5 $$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

17.6 $$B(m,n) = r^n(r+1)^m \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(r+t)^{m+n}}\,dt$$

18 BASIC DIFFERENTIAL EQUATIONS and SOLUTIONS

DIFFERENTIAL EQUATION	SOLUTION
18.1 Separation of variables $f_1(x)\, g_1(y)\, dx + f_2(x)\, g_2(y)\, dy = 0$	$\displaystyle\int \frac{f_1(x)}{f_2(x)}\, dx + \int \frac{g_2(y)}{g_1(y)}\, dy = c$
18.2 Linear first order equation $\dfrac{dy}{dx} + P(x)y = Q(x)$	$y e^{\int P\,dx} = \displaystyle\int Q e^{\int P\,dx}\, dx + c$
18.3 Bernoulli's equation $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$	$v e^{(1-n)\int P\,dx} = (1-n) \displaystyle\int Q e^{(1-n)\int P\,dx}\, dx + c$ where $v = y^{1-n}$. If $n = 1$, the solution is $\ln y = \displaystyle\int (Q - P)\, dx + c$
18.4 Exact equation $M(x,y)\, dx + N(x,y)\, dy = 0$ where $\partial M/\partial y = \partial N/\partial x$.	$\displaystyle\int M\, \partial x + \int \left(N - \frac{\partial}{\partial y}\int M\, \partial x\right) dy = c$ where ∂x indicates that the integration is to be performed with respect to x keeping y constant.
18.5 Homogeneous equation $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$	$\ln x = \displaystyle\int \frac{dv}{F(v) - v} + c$ where $v = y/x$. If $F(v) = v$, the solution is $y = cx$.

DIFFERENTIAL EQUATION	SOLUTION
18.6 $y\,F(xy)\,dx + x\,G(xy)\,dy = 0$	$\ln x = \int \frac{G(v)\,dv}{v\{G(v)-F(v)\}} + c$ where $v = xy$. If $G(v) = F(v)$, the solution is $xy = c$.
18.7 Linear, homogeneous second order equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ a, b are real constants.	Let m_1, m_2 be the roots of $m^2 + am + b = 0$. Then there are 3 cases. **Case 1.** m_1, m_2 real and distinct: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ **Case 2.** m_1, m_2 real and equal: $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$ **Case 3.** $m_1 = p + qi$, $m_2 = p - qi$: $y = e^{px}(c_1 \cos qx + c_2 \sin qx)$ where $p = -a/2$, $q = \sqrt{b - a^2/4}$.
18.8 Linear, nonhomogeneous second order equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R(x)$ a, b are real constants.	There are 3 cases corresponding to those of entry 18.7 above. **Case 1.** $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{e^{m_1 x}}{m_1 - m_2}\int e^{-m_1 x} R(x)\,dx + \frac{e^{m_2 x}}{m_2 - m_1}\int e^{-m_2 x} R(x)\,dx$ **Case 2.** $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + x e^{m_1 x}\int e^{-m_1 x} R(x)\,dx - e^{m_1 x}\int x e^{-m_1 x} R(x)\,dx$ **Case 3.** $y = e^{px}(c_1 \cos qx + c_2 \sin qx) + \frac{e^{px}\sin qx}{q}\int e^{-px} R(x) \cos qx\,dx - \frac{e^{px}\cos qx}{q}\int e^{-px} R(x) \sin qx\,dx$
18.9 Euler or Cauchy equation $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = S(x)$	Putting $x = e^t$, the equation becomes $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = S(e^t)$ and can then be solved as in entries 18.7 and 18.8 above.

DIFFERENTIAL EQUATION	SOLUTION
18.10 Bessel's equation	
$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0$	$y = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x)$ **See pages 136-137.**
18.11 Transformed Bessel's equation	
$x^2 \frac{d^2y}{dx^2} + (2p+1)x\frac{dy}{dx} + (\alpha^2x^{2r} + \beta^2)y = 0$	$y = x^{-p}\left\{c_1 J_{q/r}\left(\frac{\alpha}{r}x^r\right) + c_2 Y_{q/r}\left(\frac{\alpha}{r}x^r\right)\right\}$ where $q = \sqrt{p^2 - \beta^2}$.
18.12 Legendre's equation	
$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$	$y = c_1 P_n(x) + c_2 Q_n(x)$ **See pages 146-148.**

19 SERIES of CONSTANTS

ARITHMETIC SERIES

19.1 $$a + (a+d) + (a+2d) + \cdots + \{a + (n-1)d\} = \tfrac{1}{2}n\{2a + (n-1)d\} = \tfrac{1}{2}n(a+l)$$

where $l = a + (n-1)d$ is the last term.

Some special cases are

19.2 $$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

19.3 $$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

GEOMETRIC SERIES

19.4 $$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a-rl}{1-r}$$

where $l = ar^{n-1}$ is the last term and $r \neq 1$.

If $-1 < r < 1$, then

19.5 $$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

ARITHMETIC-GEOMETRIC SERIES

19.6 $$a + (a+d)r + (a+2d)r^2 + \cdots + \{a + (n-1)d\}r^{n-1} = \frac{a(1-r^n)}{1-r} + \frac{rd\{1 - nr^{n-1} + (n-1)r^n\}}{(1-r)^2}$$

where $r \neq 1$.

If $-1 < r < 1$, then

19.7 $$a + (a+d)r + (a+2d)r^2 + \cdots = \frac{a}{1-r} + \frac{rd}{(1-r)^2}$$

SUMS OF POWERS OF POSITIVE INTEGERS

19.8 $$1^p + 2^p + 3^p + \cdots + n^p = \frac{n^{p+1}}{p+1} + \tfrac{1}{2}n^p + \frac{B_1pn^{p-1}}{2!} - \frac{B_2p(p-1)(p-2)n^{p-3}}{4!} + \cdots$$

where the series terminates at n^2 or n according as p is odd or even, and B_k are the *Bernoulli numbers* [see page 114].

Some special cases are

19.9 $$1 + 2 + 3 + \cdots + n \;=\; \frac{n(n+1)}{2}$$

19.10 $$1^2 + 2^2 + 3^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

19.11 $$1^3 + 2^3 + 3^3 + \cdots + n^3 \;=\; \frac{n^2(n+1)^2}{4} \;=\; (1+2+3+\cdots+n)^2$$

19.12 $$1^4 + 2^4 + 3^4 + \cdots + n^4 \;=\; \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

If $S_k = 1^k + 2^k + 3^k + \cdots + n^k$ where k and n are positive integers, then

19.13 $$\binom{k+1}{1}S_1 + \binom{k+1}{2}S_2 + \cdots + \binom{k+1}{k}S_k \;=\; (n+1)^{k+1} - (n+1)$$

SERIES INVOLVING RECIPROCALS OF POWERS OF POSITIVE INTEGERS

19.14 $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \;=\; \ln 2$$

19.15 $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \;=\; \frac{\pi}{4}$$

19.16 $$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \cdots \;=\; \frac{\pi\sqrt{3}}{9} + \frac{1}{3}\ln 2$$

19.17 $$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \cdots \;=\; \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}\ln(1+\sqrt{2})}{4}$$

19.18 $$\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} - \cdots \;=\; \frac{\pi\sqrt{3}}{9} - \frac{1}{3}\ln 2$$

19.19 $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \;=\; \frac{\pi^2}{6}$$

19.20 $$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \;=\; \frac{\pi^4}{90}$$

19.21 $$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots \;=\; \frac{\pi^6}{945}$$

19.22 $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \;=\; \frac{\pi^2}{12}$$

19.23 $$\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots \;=\; \frac{7\pi^4}{720}$$

19.24 $$\frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \cdots \;=\; \frac{31\pi^6}{30{,}240}$$

19.25 $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \;=\; \frac{\pi^2}{8}$$

19.26 $$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots \;=\; \frac{\pi^4}{96}$$

19.27 $$\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots \;=\; \frac{\pi^6}{960}$$

19.28 $$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \;=\; \frac{\pi^3}{32}$$

19.29 $$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \cdots \;=\; \frac{3\pi^3\sqrt{2}}{128}$$

19.30 $$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \cdots \;=\; \frac{1}{2}$$

19.31 $$\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 6} + \cdots \;=\; \frac{3}{4}$$

19.32 $$\frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \frac{1}{7^2\cdot 9^2} + \cdots \;=\; \frac{\pi^2-8}{16}$$

19.33 $$\frac{1}{1^2\cdot 2^2\cdot 3^2}+\frac{1}{2^2\cdot 3^2\cdot 4^2}+\frac{1}{3^2\cdot 4^2\cdot 5^2}+\cdots = \frac{4\pi^2-39}{16}$$

19.34 $$\frac{1}{a}-\frac{1}{a+d}+\frac{1}{a+2d}-\frac{1}{a+3d}+\cdots = \int_0^1 \frac{u^{a-1}\,du}{1+u^d}$$

19.35 $$\frac{1}{1^{2p}}+\frac{1}{2^{2p}}+\frac{1}{3^{2p}}+\frac{1}{4^{2p}}+\cdots = \frac{2^{2p-1}\pi^{2p}B_p}{(2p)!}$$

19.36 $$\frac{1}{1^{2p}}+\frac{1}{3^{2p}}+\frac{1}{5^{2p}}+\frac{1}{7^{2p}}+\cdots = \frac{(2^{2p}-1)\pi^{2p}B_p}{2(2p)!}$$

19.37 $$\frac{1}{1^{2p}}-\frac{1}{2^{2p}}+\frac{1}{3^{2p}}-\frac{1}{4^{2p}}+\cdots = \frac{(2^{2p-1}-1)\pi^{2p}B_p}{(2p)!}$$

19.38 $$\frac{1}{1^{2p+1}}-\frac{1}{3^{2p+1}}+\frac{1}{5^{2p+1}}-\frac{1}{7^{2p+1}}+\cdots = \frac{\pi^{2p+1}E_p}{2^{2p+2}(2p)!}$$

MISCELLANEOUS SERIES

19.39 $$\frac{1}{2}+\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha = \frac{\sin(n+\frac{1}{2})\alpha}{2\sin(\alpha/2)}$$

19.40 $$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\cdots+\sin n\alpha = \frac{\sin[\frac{1}{2}(n+1)]\alpha\,\sin\frac{1}{2}n\alpha}{\sin(\alpha/2)}$$

19.41 $$1+r\cos\alpha+r^2\cos 2\alpha+r^3\cos 3\alpha+\cdots = \frac{1-r\cos\alpha}{1-2r\cos\alpha+r^2},\qquad |r|<1$$

19.42 $$r\sin\alpha+r^2\sin 2\alpha+r^3\sin 3\alpha+\cdots = \frac{r\sin\alpha}{1-2r\cos\alpha+r^2},\qquad |r|<1$$

19.43 $$1+r\cos\alpha+r^2\cos 2\alpha+\cdots+r^n\cos n\alpha = \frac{r^{n+2}\cos n\alpha-r^{n+1}\cos(n+1)\alpha-r\cos\alpha+1}{1-2r\cos\alpha+r^2}$$

19.44 $$r\sin\alpha+r^2\sin 2\alpha+\cdots+r^n\sin n\alpha = \frac{r\sin\alpha-r^{n+1}\sin(n+1)\alpha+r^{n+2}\sin n\alpha}{1-2r\cos\alpha+r^2}$$

THE EULER-MACLAURIN SUMMATION FORMULA

19.45 $$\begin{aligned}\sum_{k=1}^{n-1}F(k) &= \int_0^n F(k)\,dk-\frac{1}{2}\{F(0)+F(n)\}\\ &\quad+\frac{1}{12}\{F'(n)-F'(0)\}-\frac{1}{720}\{F'''(n)-F'''(0)\}\\ &\quad+\frac{1}{30{,}240}\{F^{(v)}(n)-F^{(v)}(0)\}-\frac{1}{1{,}209{,}600}\{F^{(vii)}(n)-F^{(vii)}(0)\}\\ &\quad+\cdots(-1)^{p-1}\frac{B_p}{(2p)!}\{F^{(2p-1)}(n)-F^{(2p-1)}(0)\}+\cdots\end{aligned}$$

THE POISSON SUMMATION FORMULA

19.46 $$\sum_{k=-\infty}^{\infty}F(k) = \sum_{m=-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}e^{2\pi imx}F(x)\,dx\right\}$$

20 TAYLOR SERIES

TAYLOR SERIES FOR FUNCTIONS OF ONE VARIABLE

20.1 $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n$$

where R_n, the remainder after n terms, is given by either of the following forms:

20.2 Lagrange's form $$R_n = \frac{f^{(n)}(\xi)(x-a)^n}{n!}$$

20.3 Cauchy's form $$R_n = \frac{f^{(n)}(\xi)(x-\xi)^{n-1}(x-a)}{(n-1)!}$$

The value ξ, which may be different in the two forms, lies between a and x. The result holds if $f(x)$ has continuous derivatives of order n at least.

If $\lim\limits_{n\to\infty} R_n = 0$, the infinite series obtained is called the *Taylor series* for $f(x)$ about $x = a$. If $a = 0$ the series is often called a *Maclaurin series*. These series, often called power series, generally converge for all values of x in some interval called the *interval of convergence* and diverge for all x outside this interval.

BINOMIAL SERIES

20.4 $$\begin{aligned}(a+x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots \\ &= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots\end{aligned}$$

Special cases are

20.5 $(a+x)^2 = a^2 + 2ax + x^2$

20.6 $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

20.7 $(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$

20.8 $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad -1 < x < 1$

20.9 $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \qquad -1 < x < 1$

20.10 $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots \qquad -1 < x < 1$

20.11 $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots \qquad -1 < x \leqq 1$

20.12 $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \cdots \qquad -1 < x \leqq 1$

20.13 $(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots \qquad -1 < x \leqq 1$

20.14 $(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \cdots \qquad -1 < x \leqq 1$

SERIES FOR EXPONENTIAL AND LOGARITHMIC FUNCTIONS

20.15 $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad -\infty < x < \infty$

20.16 $a^x = e^{x \ln a} = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \cdots \qquad -\infty < x < \infty$

20.17 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leqq 1$

20.18 $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad -1 < x < 1$

20.19 $\ln x = 2\left\{\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right\} \qquad x > 0$

20.20 $\ln x = \left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad x \geqq \frac{1}{2}$

SERIES FOR TRIGONOMETRIC FUNCTIONS

20.21 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$

20.22 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$

20.23 $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$

20.24 $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{2^{2n}B_n x^{2n-1}}{(2n)!} - \cdots \qquad 0 < |x| < \pi$

20.25 $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$

20.26 $\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15{,}120} + \cdots + \frac{2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$

20.27 $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \qquad |x| < 1$

20.28 $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \cdots\right) \qquad |x| < 1$

20.29 $\tan^{-1} x = \begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & |x| < 1 \\ \pm\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & [+ \text{ if } x \geqq 1,\ - \text{ if } x \leqq -1] \end{cases}$

20.30 $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x = \begin{cases} \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) & |x| < 1 \\ p\pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots & [p = 0 \text{ if } x > 1,\ p = 1 \text{ if } x < -1] \end{cases}$

20.31 $\sec^{-1} x = \cos^{-1}(1/x) = \frac{\pi}{2} - \left(\frac{1}{x} + \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} + \cdots\right) \qquad |x| > 1$

20.32 $\csc^{-1} x = \sin^{-1}(1/x) = \frac{1}{x} + \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} + \cdots \qquad |x| > 1$

SERIES FOR HYPERBOLIC FUNCTIONS

20.33 $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$

20.34 $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$

20.35 $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$

20.36 $\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots \frac{(-1)^{n-1}2^{2n}B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$

20.37 $\operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots \frac{(-1)^n E_n x^{2n}}{(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$

20.38 $\operatorname{csch} x = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15{,}120} + \cdots \frac{(-1)^n 2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$

20.39
$$\sinh^{-1} x = \begin{cases} x - \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5} - \dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots & |x| < 1 \\ \pm\left(\ln|2x| + \dfrac{1}{2\cdot 2x^2} - \dfrac{1\cdot 3}{2\cdot 4\cdot 4x^4} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6} - \cdots\right) & \begin{bmatrix} + \text{ if } x \geqq 1 \\ - \text{ if } x \leqq -1 \end{bmatrix} \end{cases}$$

20.40
$$\cosh^{-1} x = \pm\left\{\ln(2x) - \left(\frac{1}{2\cdot 2x^2} + \frac{1\cdot 3}{2\cdot 4\cdot 4x^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6} + \cdots\right)\right\} \qquad \begin{bmatrix} + \text{ if } \cosh^{-1} x > 0,\ x \geqq 1 \\ - \text{ if } \cosh^{-1} x < 0,\ x \geqq 1 \end{bmatrix}$$

20.41 $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad |x| < 1$

20.42 $\coth^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots \qquad |x| > 1$

MISCELLANEOUS SERIES

20.43 $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \cdots \qquad -\infty < x < \infty$

20.44 $e^{\cos x} = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31x^6}{720} + \cdots\right) \qquad -\infty < x < \infty$

20.45 $e^{\tan x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \cdots \qquad |x| < \frac{\pi}{2}$

20.46 $e^x \sin x = x + x^2 + \frac{2x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \cdots + \frac{2^{n/2}\sin(n\pi/4)\,x^n}{n!} + \cdots \qquad -\infty < x < \infty$

20.47 $e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots + \frac{2^{n/2}\cos(n\pi/4)\,x^n}{n!} + \cdots \qquad -\infty < x < \infty$

20.48 $\ln|\sin x| = \ln|x| - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots - \frac{2^{2n-1}B_n x^{2n}}{n(2n)!} + \cdots \qquad 0 < |x| < \pi$

20.49 $\ln|\cos x| = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots - \frac{2^{2n-1}(2^{2n}-1)B_n x^{2n}}{n(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$

20.50 $\ln|\tan x| = \ln|x| + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots + \frac{2^{2n}(2^{2n-1}-1)B_n x^{2n}}{n(2n)!} + \cdots \qquad 0 < |x| < \frac{\pi}{2}$

20.51 $\frac{\ln(1+x)}{1+x} = x - (1+\frac{1}{2})x^2 + (1+\frac{1}{2}+\frac{1}{3})x^3 - \cdots \qquad |x| < 1$

REVERSION OF POWER SERIES

If

20.52 $y = c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots$

then

20.53 $x = C_1y + C_2y^2 + C_3y^3 + C_4y^4 + C_5y^5 + C_6y^6 + \cdots$

where

20.54 $c_1C_1 = 1$

20.55 $c_1^3C_2 = -c_2$

20.56 $c_1^5C_3 = 2c_2^2 - c_1c_3$

20.57 $c_1^7C_4 = 5c_1c_2c_3 - 5c_2^3 - c_1^2c_4$

20.58 $c_1^9C_5 = 6c_1^2c_2c_4 + 3c_1^2c_3^2 - c_1^3c_5 + 14c_2^4 - 21c_1c_2^2c_3$

20.59 $c_1^{11}C_6 = 7c_1^3c_2c_5 + 84c_1c_2^3c_3 + 7c_1^3c_3c_4 - 28c_1^2c_2c_3^2 - c_1^4c_6 - 28c_1^2c_2^2c_4 - 42c_2^5$

TAYLOR SERIES FOR FUNCTIONS OF TWO VARIABLES

20.60
$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\{(x-a)^2f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2f_{yy}(a,b)\} + \cdots$$

where $f_x(a,b), f_y(a,b), \ldots$ denote partial derivatives with respect to $x, y, \ldots$ evaluated at $x = a,\ y = b$.

21 BERNOULLI and EULER NUMBERS

DEFINITION OF BERNOULLI NUMBERS

The *Bernoulli numbers* $B_1, B_2, B_3, \ldots$ are defined by the series

21.1 $$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} - \cdots \qquad |x| < 2\pi$$

21.2 $$1 - \frac{x}{2}\cot\frac{x}{2} = \frac{B_1 x^2}{2!} + \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} + \cdots \qquad |x| < \pi$$

DEFINITION OF EULER NUMBERS

The *Euler numbers* $E_1, E_2, E_3, \ldots$ are defined by the series

21.3 $$\operatorname{sech} x = 1 - \frac{E_1 x^2}{2!} + \frac{E_2 x^4}{4!} - \frac{E_3 x^6}{6!} + \cdots \qquad |x| < \frac{\pi}{2}$$

21.4 $$\sec x = 1 + \frac{E_1 x^2}{2!} + \frac{E_2 x^4}{4!} + \frac{E_3 x^6}{6!} + \cdots \qquad |x| < \frac{\pi}{2}$$

TABLE OF FIRST FEW BERNOULLI AND EULER NUMBERS

Bernoulli numbers	Euler numbers
$B_1 = 1/6$	$E_1 = 1$
$B_2 = 1/30$	$E_2 = 5$
$B_3 = 1/42$	$E_3 = 61$
$B_4 = 1/30$	$E_4 = 1385$
$B_5 = 5/66$	$E_5 = 50{,}521$
$B_6 = 691/2730$	$E_6 = 2{,}702{,}765$
$B_7 = 7/6$	$E_7 = 199{,}360{,}981$
$B_8 = 3617/510$	$E_8 = 19{,}391{,}512{,}145$
$B_9 = 43{,}867/798$	$E_9 = 2{,}404{,}879{,}675{,}441$
$B_{10} = 174{,}611/330$	$E_{10} = 370{,}371{,}188{,}237{,}525$
$B_{11} = 854{,}513/138$	$E_{11} = 69{,}348{,}874{,}393{,}137{,}901$
$B_{12} = 236{,}364{,}091/2730$	$E_{12} = 15{,}514{,}534{,}163{,}557{,}086{,}905$

RELATIONSHIPS OF BERNOULLI AND EULER NUMBERS

21.5 $$\binom{2n+1}{2}2^2B_1 - \binom{2n+1}{4}2^4B_2 + \binom{2n+1}{6}2^6B_3 - \cdots (-1)^{n-1}(2n+1)2^{2n}B_n = 2n$$

21.6 $$E_n = \binom{2n}{2}E_{n-1} - \binom{2n}{4}E_{n-2} + \binom{2n}{6}E_{n-3} - \cdots (-1)^n$$

21.7 $$B_n = \frac{2n}{2^{2n}(2^{2n}-1)}\left\{\binom{2n-1}{1}E_{n-1} - \binom{2n-1}{3}E_{n-2} + \binom{2n-1}{5}E_{n-3} - \cdots (-1)^{n-1}\right\}$$

SERIES INVOLVING BERNOULLI AND EULER NUMBERS

21.8 $$B_n = \frac{(2n)!}{2^{2n-1}\pi^{2n}}\left\{1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots\right\}$$

21.9 $$B_n = \frac{2(2n)!}{(2^{2n}-1)\pi^{2n}}\left\{1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \cdots\right\}$$

21.10 $$B_n = \frac{(2n)!}{(2^{2n-1}-1)\pi^{2n}}\left\{1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \cdots\right\}$$

21.11 $$E_n = \frac{2^{2n+2}(2n)!}{\pi^{2n+1}}\left\{1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \cdots\right\}$$

ASYMPTOTIC FORMULA FOR BERNOULLI NUMBERS

21.12 $$B_n \sim 4n^{2n}(\pi e)^{-2n}\sqrt{\pi n}$$

22 FORMULAS from VECTOR ANALYSIS

VECTORS AND SCALARS

Various quantities in physics such as temperature, volume and speed can be specified by a real number. Such quantities are called *scalars*.

Other quantities such as force, velocity and momentum require for their specification a direction as well as magnitude. Such quantities are called *vectors*. A vector is represented by an arrow or directed line segment indicating direction. The magnitude of the vector is determined by the length of the arrow, using an appropriate unit.

NOTATION FOR VECTORS

A vector is denoted by a bold faced letter such as $\mathbf{A}$ [Fig. 22-1]. The magnitude is denoted by $|\mathbf{A}|$ or A. The tail end of the arrow is called the *initial point* while the head is called the *terminal point*.

FUNDAMENTAL DEFINITIONS

1. **Equality of vectors.** Two vectors are equal if they have the same magnitude and direction. Thus $\mathbf{A} = \mathbf{B}$ in Fig. 22-1.

2. **Multiplication of a vector by a scalar.** If m is any real number (scalar), then $m\mathbf{A}$ is a vector whose magnitude is $|m|$ times the magnitude of $\mathbf{A}$ and whose direction is the same as or opposite to $\mathbf{A}$ according as $m > 0$ or $m < 0$. If $m = 0$, then $m\mathbf{A} = 0$ is called the *zero* or *null vector*.

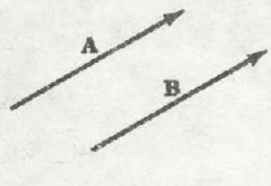


Fig. 22-1

3. **Sums of vectors.** The sum or resultant of $\mathbf{A}$ and $\mathbf{B}$ is a vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$ formed by placing the initial point of $\mathbf{B}$ on the terminal point of $\mathbf{A}$ and joining the initial point of $\mathbf{A}$ to the terminal point of $\mathbf{B}$ [Fig. 22-2(*b*)]. This definition is equivalent to the parallelogram law for vector addition as indicated in Fig. 22-2(*c*). The vector $\mathbf{A} - \mathbf{B}$ is defined as $\mathbf{A} + (-\mathbf{B})$.

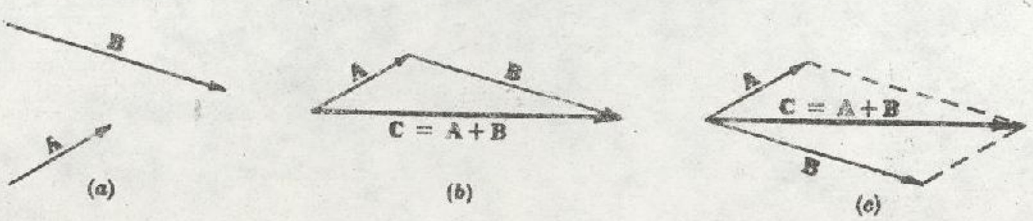
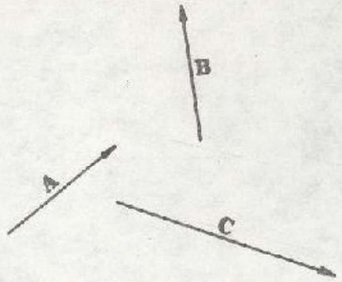
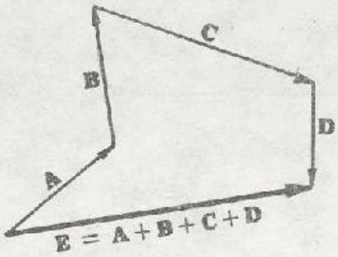


Fig. 22-2

Extensions to sums of more than two vectors are immediate. Thus Fig. 22-3 shows how to obtain the sum **E** of the vectors **A**, **B**, **C** and **D**.



(a)



(b)

Fig. 22-3

4. **Unit vectors.** A *unit vector* is a vector with unit magnitude. If **A** is a vector, then a unit vector in the direction of **A** is $\mathbf{a} = \mathbf{A}/A$ where $A > 0$.

LAWS OF VECTOR ALGEBRA

If **A**, **B**, **C** are vectors and m, n are scalars, then

22.1 $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ — Commutative law for addition

22.2 $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ — Associative law for addition

22.3 $m(n\mathbf{A}) = (mn)\mathbf{A} = n(m\mathbf{A})$ — Associative law for scalar multiplication

22.4 $(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$ — Distributive law

22.5 $m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}$ — Distributive law

COMPONENTS OF A VECTOR

A vector **A** can be represented with initial point at the origin of a rectangular coordinate system. If **i**, **j**, **k** are unit vectors in the directions of the positive x, y, z axes, then

22.6
$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

where $A_1\mathbf{i}, A_2\mathbf{j}, A_3\mathbf{k}$ are called *component vectors* of **A** in the **i**, **j**, **k** directions and A_1, A_2, A_3 are called the *components* of **A**.

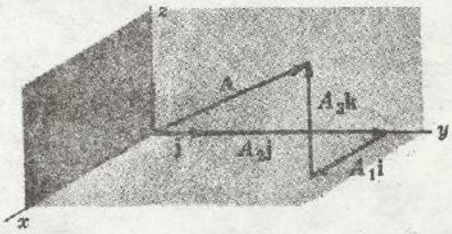


Fig. 22-4

DOT OR SCALAR PRODUCT

22.7
$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \qquad 0 \leqq \theta \leqq \pi$$

where θ is the angle between **A** and **B**.

Fundamental results are

22.8 $$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$ Commutative law

22.9 $$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$ Distributive law

22.10 $$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$.

CROSS OR VECTOR PRODUCT

22.11 $$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u} \qquad 0 \leqq \theta \leqq \pi$$

where θ is the angle between **A** and **B** and **u** is a unit vector perpendicular to the plane of **A** and **B** such that **A, B, u** form a *right-handed system* [i.e. a right-threaded screw rotated through an angle less than 180° from **A** to **B** will advance in the direction of **u** as in Fig. 22-5].

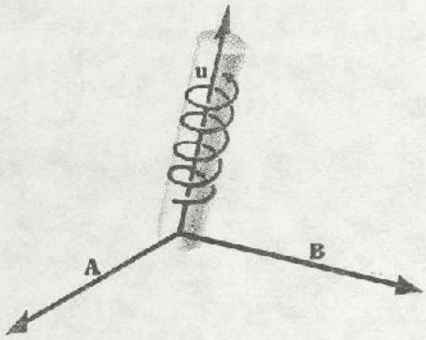


Fig. 22-5

Fundamental results are

22.12 $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$
$$= (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}$$

22.13 $$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

22.14 $$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

22.15 $|\mathbf{A}\times\mathbf{B}|$ = area of parallelogram having sides **A** and **B**

MISCELLANEOUS FORMULAS INVOLVING DOT AND CROSS PRODUCTS

22.16 $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2$$

22.17 $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ = volume of parallelepiped with sides **A, B, C**

22.18 $$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

22.19 $$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$$

22.20 $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

22.21 $$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D})\} - \mathbf{D}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\}$$
$$= \mathbf{B}\{\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})\} - \mathbf{A}\{\mathbf{B}\cdot(\mathbf{C}\times\mathbf{D})\}$$

DERIVATIVES OF VECTORS

The derivative of a vector function $\mathbf{A}(u) = A_1(u)\mathbf{i} + A_2(u)\mathbf{j} + A_3(u)\mathbf{k}$ of the scalar variable u is given by

22.22 $$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u+\Delta u) - \mathbf{A}(u)}{\Delta u} = \frac{dA_1}{du}\mathbf{i} + \frac{dA_2}{du}\mathbf{j} + \frac{dA_3}{du}\mathbf{k}$$

Partial derivatives of a vector function $\mathbf{A}(x, y, z)$ are similarly defined. We assume that all derivatives exist unless otherwise specified.

FORMULAS INVOLVING DERIVATIVES

22.23 $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

22.24 $$\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

22.25 $$\frac{d}{du}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\} = \frac{d\mathbf{A}}{du}\cdot(\mathbf{B}\times\mathbf{C}) + \mathbf{A}\cdot\left(\frac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \mathbf{A}\cdot\left(\mathbf{B}\times\frac{d\mathbf{C}}{du}\right)$$

22.26 $$\mathbf{A}\cdot\frac{d\mathbf{A}}{du} = A\frac{dA}{du}$$

22.27 $$\mathbf{A}\cdot\frac{d\mathbf{A}}{du} = 0 \quad \text{if } |\mathbf{A}| \text{ is a constant}$$

THE DEL OPERATOR

The operator *del* is defined by

22.28 $$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

In the results below we assume that $U = U(x, y, z)$, $V = V(x, y, z)$, $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ have partial derivatives.

THE GRADIENT

22.29 $$\text{Gradient of } U = \text{grad } U = \nabla U = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}$$

THE DIVERGENCE

22.30 $$\begin{aligned}\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \nabla\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned}$$

THE CURL

22.31 Curl of $\mathbf{A}$ = curl $\mathbf{A}$ = $\nabla \times \mathbf{A}$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

THE LAPLACIAN

22.32 Laplacian of U = $\nabla^2 U = \nabla \cdot (\nabla U) = \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2}$

22.33 Laplacian of $\mathbf{A}$ = $\nabla^2 \mathbf{A} = \dfrac{\partial^2 \mathbf{A}}{\partial x^2} + \dfrac{\partial^2 \mathbf{A}}{\partial y^2} + \dfrac{\partial^2 \mathbf{A}}{\partial z^2}$

THE BIHARMONIC OPERATOR

22.34 Biharmonic operator on U = $\nabla^4 U = \nabla^2(\nabla^2 U)$

$$= \frac{\partial^4 U}{\partial x^4} + \frac{\partial^4 U}{\partial y^4} + \frac{\partial^4 U}{\partial z^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + 2\frac{\partial^4 U}{\partial y^2 \partial z^2} + 2\frac{\partial^4 U}{\partial x^2 \partial z^2}$$

MISCELLANEOUS FORMULAS INVOLVING ∇

22.35 $\nabla(U+V) = \nabla U + \nabla V$

22.36 $\nabla \cdot (\mathbf{A}+\mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

22.37 $\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

22.38 $\nabla \cdot (U\mathbf{A}) = (\nabla U) \cdot \mathbf{A} + U(\nabla \cdot \mathbf{A})$

22.39 $\nabla \times (U\mathbf{A}) = (\nabla U) \times \mathbf{A} + U(\nabla \times \mathbf{A})$

22.40 $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

22.41 $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$

22.42 $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$

22.43 $\nabla \times (\nabla U) = 0$, i.e. the curl of the gradient of U is zero.

22.44 $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, i.e. the divergence of the curl of $\mathbf{A}$ is zero.

22.45 $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

INTEGRALS INVOLVING VECTORS

If $\mathbf{A}(u) = \dfrac{d}{du}\mathbf{B}(u)$, then the *indefinite integral* of $\mathbf{A}(u)$ is

22.46 $$\int \mathbf{A}(u)\,du \;=\; \mathbf{B}(u) + \mathbf{c} \qquad \mathbf{c} = \text{constant vector}$$

The *definite integral* of $\mathbf{A}(u)$ from $u = a$ to $u = b$ in this case is given by

22.47 $$\int_a^b \mathbf{A}(u)\,du \;=\; \mathbf{B}(b) - \mathbf{B}(a)$$

The definite integral can be defined as on page 94.

LINE INTEGRALS

Consider a space curve C joining two points $P_1(a_1, a_2, a_3)$ and $P_2(b_1, b_2, b_3)$ as in Fig. 22-6. Divide the curve into n parts by points of subdivision $(x_1, y_1, z_1), \ldots, (x_{n-1}, y_{n-1}, z_{n-1})$. Then the *line integral* of a vector $\mathbf{A}(x, y, z)$ along C is defined as

22.48 $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; \lim_{n\to\infty} \sum_{p=1}^{n} \mathbf{A}(x_p, y_p, z_p)\cdot \Delta\mathbf{r}_p$$

where $\Delta\mathbf{r}_p = \Delta x_p\,\mathbf{i} + \Delta y_p\,\mathbf{j} + \Delta z_p\,\mathbf{k}$, $\Delta x_p = x_{p+1} - x_p$, $\Delta y_p = y_{p+1} - y_p$, $\Delta z_p = z_{p+1} - z_p$ and where it is assumed that as $n \to \infty$ the largest of the magnitudes $|\Delta\mathbf{r}_p|$ approaches zero. The result 22.48 is a generalization of the ordinary definite integral [page 94].

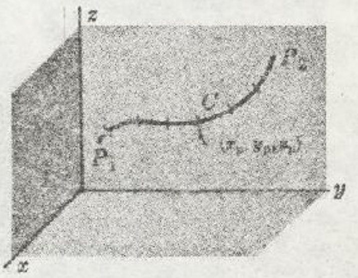


Fig. 22-6

The line integral 22.48 can also be written

22.49 $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_C (A_1\,dx + A_2\,dy + A_3\,dz)$$

using $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

PROPERTIES OF LINE INTEGRALS

22.50 $$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; -\int_{P_2}^{P_1} \mathbf{A}\cdot d\mathbf{r}$$

22.51 $$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{P_1}^{P_3} \mathbf{A}\cdot d\mathbf{r} + \int_{P_3}^{P_2} \mathbf{A}\cdot d\mathbf{r}$$

INDEPENDENCE OF THE PATH

In general a line integral has a value which depends on the particular path C joining points P_1 and P_2 in a region $\mathcal{R}$. However, in case $\mathbf{A} = \nabla\phi$ or $\nabla \times \mathbf{A} = \mathbf{0}$ where ϕ and its partial derivatives are continuous in $\mathcal{R}$, the line integral $\int_C \mathbf{A}\cdot d\mathbf{r}$ is independent of the path. In such case

22.52 $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; \phi(P_2) - \phi(P_1)$$

where $\phi(P_1)$ and $\phi(P_2)$ denote the values of ϕ at P_1 and P_2 respectively. In particular if C is a closed curve,

22.53 $$\int_C \mathbf{A} \cdot d\mathbf{r} = \oint_C \mathbf{A} \cdot d\mathbf{r} = 0$$

where the circle on the integral sign is used to emphasize that C is closed.

MULTIPLE INTEGRALS

Let $F(x, y)$ be a function defined in a region $\mathcal{R}$ of the xy plane as in Fig. 22-7. Subdivide the region into n parts by lines parallel to the x and y axes as indicated. Let $\Delta A_p = \Delta x_p \, \Delta y_p$ denote an area of one of these parts. Then the integral of $F(x, y)$ over $\mathcal{R}$ is defined as

22.54 $$\int_{\mathcal{R}} F(x, y)\, dA = \lim_{n \to \infty} \sum_{p=1}^{n} F(x_p, y_p)\, \Delta A_p$$

provided this limit exists.

In such case the integral can also be written as

22.55 $$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} F(x, y)\, dy\, dx = \int_{x=a}^{b} \left\{ \int_{y=f_1(x)}^{f_2(x)} F(x, y)\, dy \right\} dx$$

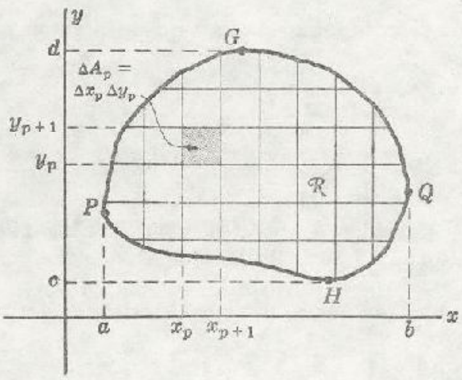


Fig. 22-7

where $y = f_1(x)$ and $y = f_2(x)$ are the equations of curves PHQ and PGQ respectively and a and b are the x coordinates of points P and Q. The result can also be written as

22.56 $$\int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} F(x, y)\, dx\, dy = \int_{y=c}^{d} \left\{ \int_{x=g_1(y)}^{g_2(y)} F(x, y)\, dx \right\} dy$$

where $x = g_1(y)$, $x = g_2(y)$ are the equations of curves HPG and HQG respectively and c and d are the y coordinates of H and G.

These are called *double integrals* or *area integrals*. The ideas can be similarly extended to *triple* or *volume integrals* or to higher *multiple integrals*.

SURFACE INTEGRALS

Subdivide the surface S [see Fig. 22-8] into n elements of area ΔS_p, $p = 1, 2, \ldots, n$. Let $\mathbf{A}(x_p, y_p, z_p) = \mathbf{A}_p$ where (x_p, y_p, z_p) is a point P in ΔS_p. Let $\mathbf{N}_p$ be a unit normal to ΔS_p at P. Then the surface integral of the normal component of $\mathbf{A}$ over S is defined as

22.57 $$\int_S \mathbf{A} \cdot \mathbf{N}\, dS = \lim_{n \to \infty} \sum_{p=1}^{n} \mathbf{A}_p \cdot \mathbf{N}_p\, \Delta S_p$$

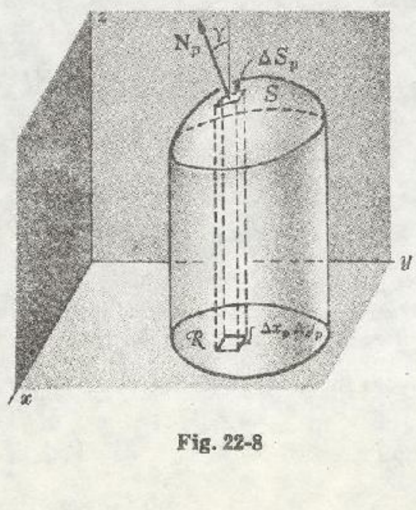


Fig. 22-8

RELATION BETWEEN SURFACE AND DOUBLE INTEGRALS

If $\mathcal{R}$ is the projection of S on the xy plane, then [see Fig. 22-8]

22.58
$$\int_S \mathbf{A}\cdot\mathbf{N}\,dS = \iint_{\mathcal{R}} \mathbf{A}\cdot\mathbf{N}\,\frac{dx\,dy}{|\mathbf{N}\cdot\mathbf{k}|}$$

THE DIVERGENCE THEOREM

Let S be a closed surface bounding a region of volume V; then if $\mathbf{N}$ is the positive (outward drawn) normal and $d\mathbf{S} = \mathbf{N}\,dS$, we have [see Fig. 22-9]

22.59
$$\int_V \nabla\cdot\mathbf{A}\,dV = \int_S \mathbf{A}\cdot d\mathbf{S}$$

The result is also called *Gauss' theorem* or *Green's theorem.*

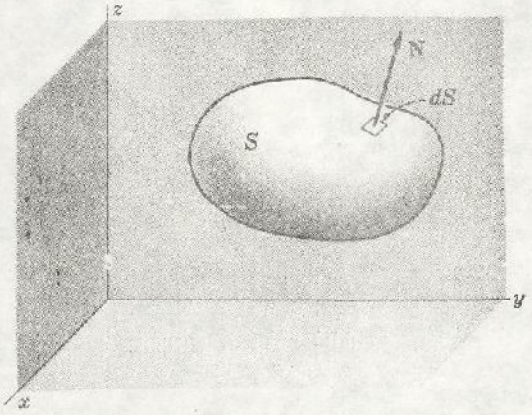


Fig. 22-9

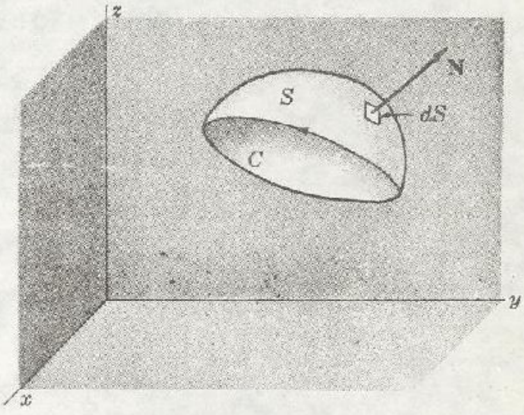


Fig. 22-10

STOKE'S THEOREM

Let S be an open two-sided surface bounded by a closed non-intersecting curve C [simple closed curve] as in Fig. 22-10. Then

22.60
$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where the circle on the integral is used to emphasize that C is closed.

GREEN'S THEOREM IN THE PLANE

22.61
$$\oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where R is the area bounded by the closed curve C. This result is a special case of the divergence theorem or Stoke's theorem.

GREEN'S FIRST IDENTITY

22.62 $$\int_V \{\phi\,\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\}\,dV \;=\; \int (\phi\,\nabla\psi)\cdot d\mathbf{S}$$

where ϕ and ψ are scalar functions.

GREEN'S SECOND IDENTITY

22.63 $$\int_V (\phi\,\nabla^2\psi - \psi\,\nabla^2\phi)\,dV \;=\; \int_S (\phi\,\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

MISCELLANEOUS INTEGRAL THEOREMS

22.64 $$\int_V \nabla\times\mathbf{A}\,dV \;=\; \int_S d\mathbf{S}\times\mathbf{A}$$

22.65 $$\int_C \phi\,d\mathbf{r} \;=\; \int_S d\mathbf{S}\times\nabla\phi$$

CURVILINEAR COORDINATES

A point P in space [see Fig. 22-11] can be located by rectangular coordinates (x, y, z) or curvilinear coordinates (u_1, u_2, u_3) where the transformation equations from one set of coordinates to the other are given by

22.66
$$x = x(u_1, u_2, u_3)$$
$$y = y(u_1, u_2, u_3)$$
$$z = z(u_1, u_2, u_3)$$

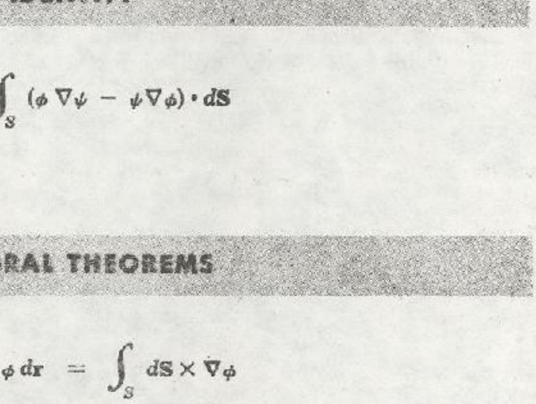


Fig. 22-11

If u_2 and u_3 are constant, then as u_1 varies, the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of P describes a curve called the u_1 coordinate curve. Similarly we define the u_2 and u_3 coordinate curves through P. The vectors $\partial\mathbf{r}/\partial u_1$, $\partial\mathbf{r}/\partial u_2$, $\partial\mathbf{r}/\partial u_3$ represent tangent vectors to the u_1, u_2, u_3 coordinate curves. Letting $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit tangent vectors to these curves, we have

22.67 $$\frac{\partial\mathbf{r}}{\partial u_1} = h_1\mathbf{e}_1, \quad \frac{\partial\mathbf{r}}{\partial u_2} = h_2\mathbf{e}_2, \quad \frac{\partial\mathbf{r}}{\partial u_3} = h_3\mathbf{e}_3$$

where

22.68 $$h_1 = \left|\frac{\partial\mathbf{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial\mathbf{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial\mathbf{r}}{\partial u_3}\right|$$

are called *scale factors*. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are mutually perpendicular, the curvilinear coordinate system is called *orthogonal*.

FORMULAS INVOLVING ORTHOGONAL CURVILINEAR COORDINATES

22.69 $$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3 = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3$$

22.70 $$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2$$

where ds is the element of arc length.

If dV is the element of volume, then

22.71 $$dV = |(h_1\mathbf{e}_1\,du_1)\cdot(h_2\mathbf{e}_2\,du_2)\times(h_3\mathbf{e}_3\,du_3)| = h_1h_2h_3\,du_1\,du_2\,du_3$$
$$= \left|\frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3}\right|du_1\,du_2\,du_3 = \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right|du_1\,du_2\,du_3$$

where

22.72 $$\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} = \begin{vmatrix} \partial x/\partial u_1 & \partial x/\partial u_2 & \partial x/\partial u_3 \\ \partial y/\partial u_1 & \partial y/\partial u_2 & \partial y/\partial u_3 \\ \partial z/\partial u_1 & \partial z/\partial u_2 & \partial z/\partial u_3 \end{vmatrix}$$

is called the *Jacobian* of the transformation.

TRANSFORMATION OF MULTIPLE INTEGRALS

The result 22.72 can be used to transform multiple integrals from rectangular to curvilinear coordinates. For example, we have

22.73 $$\iiint\limits_{\mathcal{R}} F(x,y,z)\,dx\,dy\,dz = \iiint\limits_{\mathcal{R}'} G(u_1,u_2,u_3)\left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right|du_1\,du_2\,du_3$$

where $\mathcal{R}'$ is the region into which $\mathcal{R}$ is mapped by the transformation and $G(u_1,u_2,u_3)$ is the value of $F(x,y,z)$ corresponding to the transformation.

GRADIENT, DIVERGENCE, CURL AND LAPLACIAN

In the following, Φ is a scalar function and $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3.

22.74 Gradient of Φ = grad Φ = $\nabla\Phi$ = $\dfrac{\mathbf{e}_1}{h_1}\dfrac{\partial\Phi}{\partial u_1} + \dfrac{\mathbf{e}_2}{h_2}\dfrac{\partial\Phi}{\partial u_2} + \dfrac{\mathbf{e}_3}{h_3}\dfrac{\partial\Phi}{\partial u_3}$

22.75 Divergence of $\mathbf{A}$ = div $\mathbf{A}$ = $\nabla\cdot\mathbf{A}$ = $\dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}(h_2h_3A_1) + \dfrac{\partial}{\partial u_2}(h_3h_1A_2) + \dfrac{\partial}{\partial u_3}(h_1h_2A_3)\right]$

22.76 Curl of $\mathbf{A}$ = curl $\mathbf{A}$ = $\nabla\times\mathbf{A}$ = $\dfrac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$

$$= \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\partial}{\partial u_3}(h_2A_2)\right]\mathbf{e}_1 + \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_3}(h_1A_1) - \frac{\partial}{\partial u_1}(h_3A_3)\right]\mathbf{e}_2$$
$$+ \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\mathbf{e}_3$$

22.77 Laplacian of Φ = $\nabla^2\Phi$ = $\dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial\Phi}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_3h_1}{h_2}\dfrac{\partial\Phi}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial\Phi}{\partial u_3}\right)\right]$

Note that the biharmonic operator $\nabla^4\Phi = \nabla^2(\nabla^2\Phi)$ can be obtained from 22.77.

Paraboloidal Coordinates (u, v, ϕ)

22.87 $$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \tfrac{1}{2}(u^2 - v^2)$$

where $$u \geqq 0, \quad v \geqq 0, \quad 0 \leqq \phi < 2\pi$$

22.88 $$h_1^2 = h_2^2 = u^2 + v^2, \quad h_3^2 = u^2v^2$$

22.89 $$\nabla^2\Phi = \frac{1}{u(u^2+v^2)}\frac{\partial}{\partial u}\left(u\frac{\partial\Phi}{\partial u}\right) + \frac{1}{v(u^2+v^2)}\frac{\partial}{\partial v}\left(v\frac{\partial\Phi}{\partial v}\right) + \frac{1}{u^2v^2}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the parabolas of Fig. 22-14 about the x axis which is then relabeled the z axis.

Elliptic Cylindrical Coordinates (u, v, z)

22.90 $$x = a\cosh u\cos v, \quad y = a\sinh u\sin v, \quad z = z$$

where $$u \geqq 0, \quad 0 \leqq v < 2\pi, \quad -\infty < z < \infty$$

22.91 $$h_1^2 = h_2^2 = a^2(\sinh^2 u + \sin^2 v), \quad h_3^2 = 1$$

22.92 $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-15. They are confocal ellipses and hyperbolas.

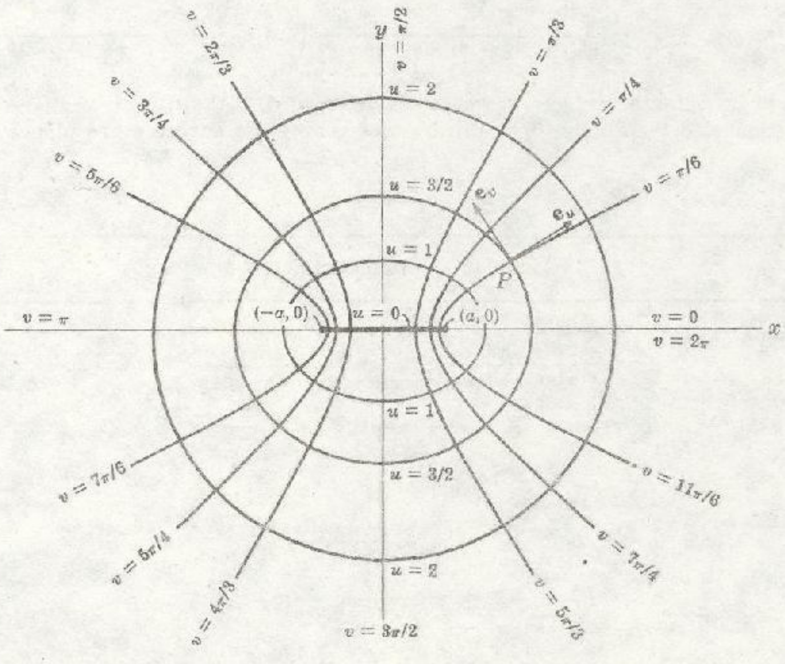


Fig. 22-15. Elliptic cylindrical coordinates.

Prolate Spheroidal Coordinates (ξ, η, ϕ)

22.93 $$x = a \sinh\xi \sin\eta \cos\phi, \quad y = a \sinh\xi \sin\eta \sin\phi, \quad z = a \cosh\xi \cos\eta$$

where $$\xi \geqq 0, \quad 0 \leqq \eta \leqq \pi, \quad 0 \leqq \phi < 2\pi$$

22.94 $$h_1^2 = h_2^2 = a^2(\sinh^2\xi + \sin^2\eta), \quad h_3^2 = a^2 \sinh^2\xi \sin^2\eta$$

22.95 $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\sinh\xi}\frac{\partial}{\partial\xi}\left(\sinh\xi\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\sinh^2\xi\sin^2\eta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the x axis which is relabeled the z axis. The third set of coordinate surfaces consists of planes passing through this axis.

Oblate Spheroidal Coordinates (ξ, η, ϕ)

22.96 $$x = a \cosh\xi \cos\eta \cos\phi, \quad y = a \cosh\xi \cos\eta \sin\phi, \quad z = a \sinh\xi \sin\eta$$

where $$\xi \geqq 0, \quad -\pi/2 \leqq \eta \leqq \pi/2, \quad 0 \leqq \phi < 2\pi$$

22.97 $$h_1^2 = h_2^2 = a^2(\sinh^2\xi + \sin^2\eta), \quad h_3^2 = a^2 \cosh^2\xi \cos^2\eta$$

22.98 $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\cosh\xi}\frac{\partial}{\partial\xi}\left(\cosh\xi\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\cos\eta}\frac{\partial}{\partial\eta}\left(\cos\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\cosh^2\xi\cos^2\eta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the y axis which is relabeled the z axis. The third set of coordinate surfaces are planes passing through this axis.

Bipolar Coordinates (u, v, z)

22.99 $$x = \frac{a \sinh v}{\cosh v - \cos u}, \quad y = \frac{a \sin u}{\cosh v - \cos u}, \quad z = z$$

where $$0 \leqq u < 2\pi, \quad -\infty < v < \infty, \quad -\infty < z < \infty$$

or

22.100 $$x^2 + (y - a\cot u)^2 = a^2\csc^2 u, \quad (x - a\coth v)^2 + y^2 = a^2\operatorname{csch}^2 v, \quad z = z$$

22.101 $$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = 1$$

22.102 $$\nabla^2\Phi = \frac{(\cosh v - \cos u)^2}{a^2}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-16 below.

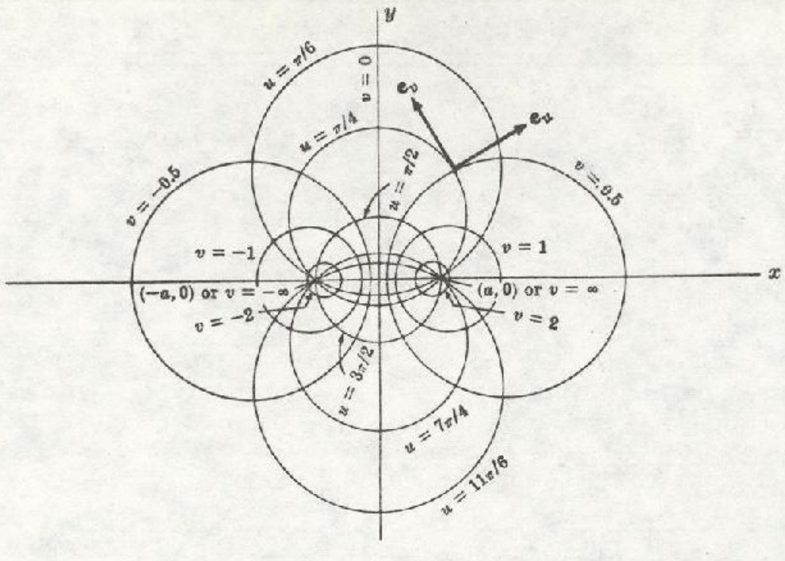


Fig. 22-16. Bipolar coordinates.

Toroidal Coordinates (u, v, ϕ)

22.103 $$x = \frac{a \sinh v \cos \phi}{\cosh v - \cos u}, \quad y = \frac{a \sinh v \sin \phi}{\cosh v - \cos u}, \quad z = \frac{a \sin u}{\cosh v - \cos u}$$

22.104 $$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = \frac{a^2 \sinh^2 v}{(\cosh v - \cos u)^2}$$

22.105 $$\nabla^2 \Phi = \frac{(\cosh v - \cos u)^3}{a^2} \frac{\partial}{\partial u}\left(\frac{1}{\cosh v - \cos u} \frac{\partial \Phi}{\partial u}\right) + \frac{(\cosh v - \cos u)^3}{a^2 \sinh v} \frac{\partial}{\partial v}\left(\frac{\sinh v}{\cosh v - \cos u} \frac{\partial \Phi}{\partial v}\right) + \frac{(\cosh v - \cos u)^2}{a^2 \sinh^2 v} \frac{\partial^2 \Phi}{\partial \phi^2}$$

The coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the y axis which is relabeled the z axis.

Conical Coordinates (λ, μ, ν)

22.106 $$x = \frac{\lambda \mu \nu}{ab}, \quad y = \frac{\lambda}{a}\sqrt{\frac{(\mu^2 - a^2)(\nu^2 - a^2)}{a^2 - b^2}}, \quad z = \frac{\lambda}{b}\sqrt{\frac{(\mu^2 - b^2)(\nu^2 - b^2)}{b^2 - a^2}}$$

22.107 $$h_1^2 = 1, \quad h_2^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\mu^2 - a^2)(b^2 - \mu^2)}, \quad h_3^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\nu^2 - a^2)(\nu^2 - b^2)}$$

Confocal Ellipsoidal Coordinates (λ, μ, ν)

22.108 $$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} + \dfrac{z^2}{c^2-\lambda} = 1 & \lambda < c^2 < b^2 < a^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} + \dfrac{z^2}{c^2-\mu} = 1 & c^2 < \mu < b^2 < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} + \dfrac{z^2}{c^2-\nu} = 1 & c^2 < b^2 < \nu < a^2 \end{cases}$$

or

22.109 $$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)} \\ z^2 = \dfrac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(c^2-a^2)(c^2-b^2)} \end{cases}$$

22.110 $$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)(c^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{4(a^2-\nu)(b^2-\nu)(c^2-\nu)} \end{cases}$$

Confocal Paraboloidal Coordinates (λ, μ, ν)

22.111 $$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} = z - \lambda & -\infty < \lambda < b^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} = z - \mu & b^2 < \mu < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} = z - \nu & a^2 < \nu < \infty \end{cases}$$

or

22.112 $$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2} \\ z = \lambda + \mu + \nu - a^2 - b^2 \end{cases}$$

22.113 $$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{16(a^2-\nu)(b^2-\nu)} \end{cases}$$

23 FOURIER SERIES

DEFINITION OF A FOURIER SERIES

The Fourier series corresponding to a function $f(x)$ defined in the interval $c \leqq x \leqq c + 2L$ where c and $L > 0$ are constants, is defined as

23.1
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

23.2
$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_c^{c+2L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n = \dfrac{1}{L} \displaystyle\int_c^{c+2L} f(x) \sin \frac{n\pi x}{L}\, dx \end{cases}$$

If $f(x)$ and $f'(x)$ are piecewise continuous and $f(x)$ is defined by periodic extension of period $2L$, i.e. $f(x + 2L) = f(x)$, then the series converges to $f(x)$ if x is a point of continuity and to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ if x is a point of discontinuity.

COMPLEX FORM OF FOURIER SERIES

Assuming that the series 23.1 converges to $f(x)$, we have

23.3
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where

23.4
$$c_n = \frac{1}{2L} \int_c^{c+2L} f(x) e^{-in\pi x/L}\, dx = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \\ \frac{1}{2}a_0 & n = 0 \end{cases}$$

PARSEVAL'S IDENTITY

23.5
$$\frac{1}{L} \int_c^{c+2L} \{f(x)\}^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

GENERALIZED PARSEVAL IDENTITY

23.6
$$\frac{1}{L} \int_c^{c+2L} f(x)\, g(x)\, dx = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n)$$

where a_n, b_n and c_n, d_n are the Fourier coefficients corresponding to $f(x)$ and $g(x)$ respectively.

SPECIAL FOURIER SERIES AND THEIR GRAPHS

23.7 $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

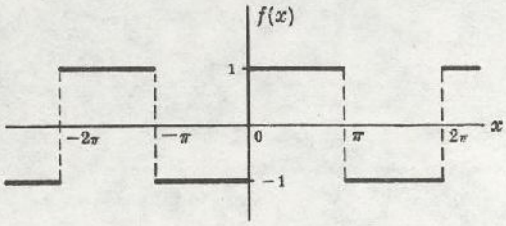


Fig. 23-1

23.8 $f(x) = |x| = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases}$

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

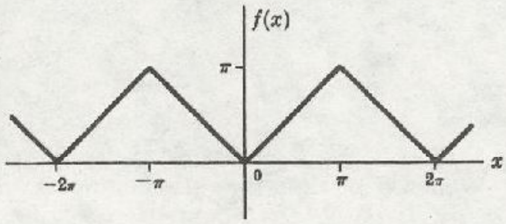


Fig. 23-2

23.9 $f(x) = x, \ -\pi < x < \pi$

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

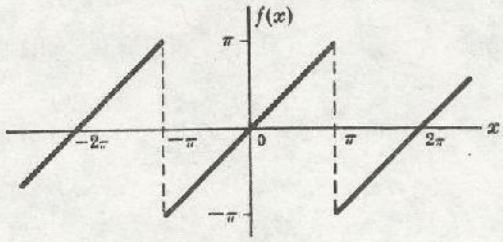


Fig. 23-3

23.10 $f(x) = x, \ 0 < x < 2\pi$

$$\pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

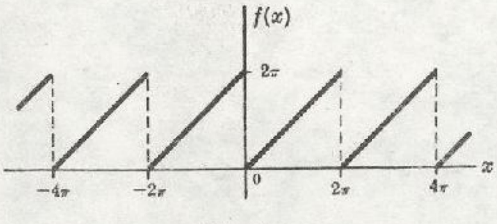


Fig. 23-4

23.11 $f(x) = |\sin x|, \ -\pi < x < \pi$

$$\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} + \cdots\right)$$

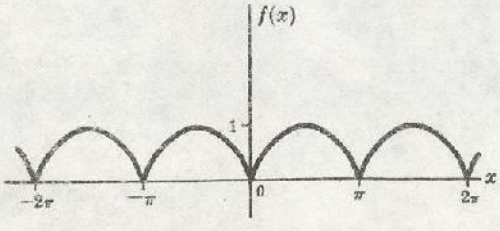


Fig. 23-5

23.12 $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$

$$\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} + \cdots\right)$$

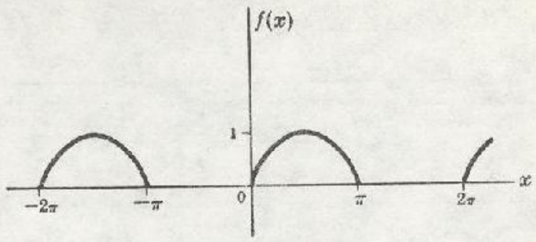


Fig. 23-6

23.13 $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ -\cos x & -\pi < x < 0 \end{cases}$

$$\frac{8}{\pi}\left(\frac{\sin 2x}{1 \cdot 3} + \frac{2 \sin 4x}{3 \cdot 5} + \frac{3 \sin 6x}{5 \cdot 7} + \cdots\right)$$

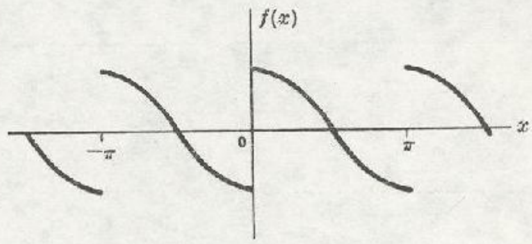


Fig. 23-7

23.14 $f(x) = x^2, \ -\pi < x < \pi$

$$\frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

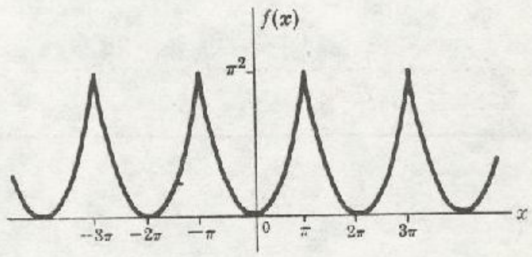


Fig. 23-8

23.15 $f(x) = x(\pi - x), \ 0 < x < \pi$

$$\frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right)$$

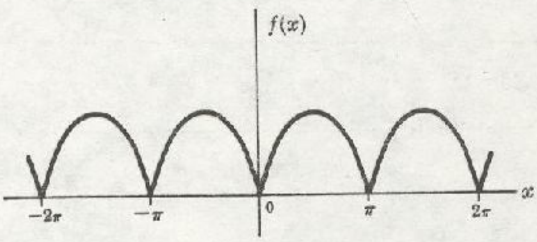


Fig. 23-9

23.16 $f(x) = x(\pi - x)(\pi + x), \ -\pi < x < \pi$

$$12\left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots\right)$$

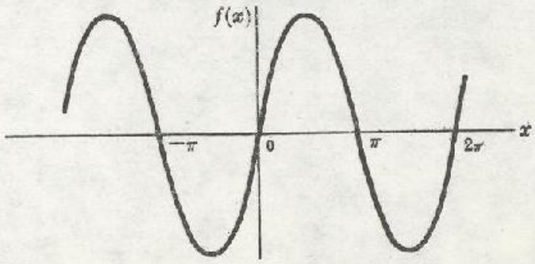


Fig. 23-10

23.17 $f(x) = \begin{cases} 0 & 0 < x < \pi - \alpha \\ 1 & \pi - \alpha < x < \pi + \alpha \\ 0 & \pi + \alpha < x < 2\pi \end{cases}$

$$\frac{\alpha}{\pi} - \frac{2}{\pi}\left(\frac{\sin\alpha\cos x}{1} - \frac{\sin 2\alpha\cos 2x}{2} + \frac{\sin 3\alpha\cos 3x}{3} - \cdots\right)$$

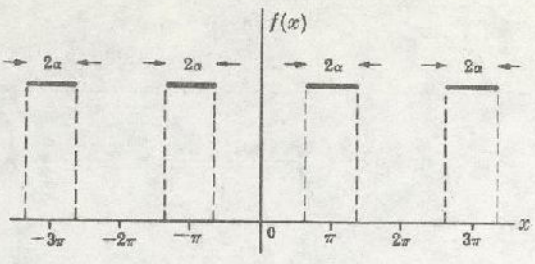


Fig. 23-11

23.18 $f(x) = \begin{cases} x(\pi - x) & 0 < x < \pi \\ -x(\pi - x) & -\pi < x < 0 \end{cases}$

$$\frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)$$

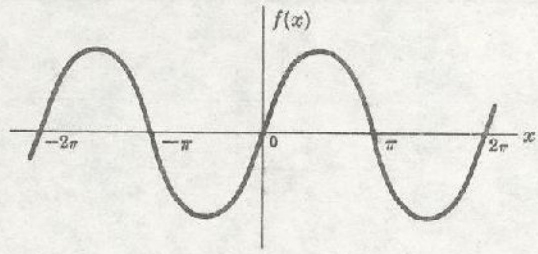


Fig. 23-12

MISCELLANEOUS FOURIER SERIES

23.19 $f(x) = \sin\mu x, \quad -\pi < x < \pi, \quad \mu \neq \text{integer}$

$$\frac{2\sin\mu\pi}{\pi}\left(\frac{\sin x}{1^2 - \mu^2} - \frac{2\sin 2x}{2^2 - \mu^2} + \frac{3\sin 3x}{3^2 - \mu^2} - \cdots\right)$$

23.20 $f(x) = \cos\mu x, \quad -\pi < x < \pi, \quad \mu \neq \text{integer}$

$$\frac{2\mu\sin\mu\pi}{\pi}\left(\frac{1}{2\mu^2} + \frac{\cos x}{1^2 - \mu^2} - \frac{\cos 2x}{2^2 - \mu^2} + \frac{\cos 3x}{3^2 - \mu^2} - \cdots\right)$$

23.21 $f(x) = \tan^{-1}[(a\sin x)/(1 - a\cos x)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a\sin x + \frac{a^2}{2}\sin 2x + \frac{a^3}{3}\sin 3x + \cdots$$

23.22 $f(x) = \ln(1 - 2a\cos x + a^2), \quad -\pi < x < \pi, \quad |a| < 1$

$$-2\left(a\cos x + \frac{a^2}{2}\cos 2x + \frac{a^3}{3}\cos 3x + \cdots\right)$$

23.23 $f(x) = \frac{1}{2}\tan^{-1}[(2a\sin x)/(1 - a^2)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a\sin x + \frac{a^3}{3}\sin 3x + \frac{a^5}{5}\sin 5x + \cdots$$

23.24 $f(x) = \frac{1}{2}\tan^{-1}[(2a\cos x)/(1-a^2)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a\cos x - \frac{a^3}{3}\cos 3x + \frac{a^5}{5}\cos 5x - \cdots$$

23.25 $f(x) = e^{\mu x}, \quad -\pi < x < \pi$

$$\frac{2\sinh\mu\pi}{\pi}\left(\frac{1}{2\mu} + \sum_{n=1}^{\infty}\frac{(-1)^n(\mu\cos nx - n\sin nx)}{\mu^2+n^2}\right)$$

23.26 $f(x) = \sinh\mu x, \quad -\pi < x < \pi$

$$\frac{2\sinh\mu\pi}{\pi}\left(\frac{\sin x}{1^2+\mu^2} - \frac{2\sin 2x}{2^2+\mu^2} + \frac{3\sin 3x}{3^2+\mu^2} - \cdots\right)$$

23.27 $f(x) = \cosh\mu x, \quad -\pi < x < \pi$

$$\frac{2\mu\sinh\mu\pi}{\pi}\left(\frac{1}{2\mu^2} - \frac{\cos x}{1^2+\mu^2} + \frac{\cos 2x}{2^2+\mu^2} - \frac{\cos 3x}{3^2+\mu^2} + \cdots\right)$$

23.28 $f(x) = \ln|\sin\frac{1}{2}x|, \quad 0 < x < \pi$

$$-\left(\ln 2 + \frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots\right)$$

23.29 $f(x) = \ln|\cos\frac{1}{2}x|, \quad -\pi < x < \pi$

$$-\left(\ln 2 - \frac{\cos x}{1} + \frac{\cos 2x}{2} - \frac{\cos 3x}{3} + \cdots\right)$$

23.30 $f(x) = \frac{1}{6}\pi^2 - \frac{1}{2}\pi x + \frac{1}{4}x^2, \quad 0 \leqq x \leqq 2\pi$

$$\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$$

23.31 $f(x) = \frac{1}{12}x(x-\pi)(x-2\pi), \quad 0 \leqq x \leqq 2\pi$

$$\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \cdots$$

23.32 $f(x) = \frac{1}{90}\pi^4 - \frac{1}{12}\pi^2x^2 + \frac{1}{12}\pi x^3 - \frac{1}{48}x^4, \quad 0 \leqq x \leqq 2\pi$

$$\frac{\cos x}{1^4} + \frac{\cos 2x}{2^4} + \frac{\cos 3x}{3^4} + \cdots$$

24 BESSEL FUNCTIONS

BESSEL'S DIFFERENTIAL EQUATION

24.1 $$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called *Bessel functions of order n*.

BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER n

24.2 $$\begin{aligned} J_n(x) &= \frac{x^n}{2^n\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(x/2)^{n+2k}}{k!\,\Gamma(n+k+1)} \end{aligned}$$

24.3 $$\begin{aligned} J_{-n}(x) &= \frac{x^{-n}}{2^{-n}\Gamma(1-n)}\left\{1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2\cdot 4(2-2n)(4-2n)} - \cdots\right\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(x/2)^{2k-n}}{k!\,\Gamma(k+1-n)} \end{aligned}$$

24.4 $$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 0,1,2,\ldots$$

If $n \neq 0,1,2,\ldots,$ $J_n(x)$ and $J_{-n}(x)$ are linearly independent.

If $n \neq 0,1,2,\ldots,$ $J_n(x)$ is bounded at $x = 0$ while $J_{-n}(x)$ is unbounded.

For $n = 0,1$ we have

24.5 $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

24.6 $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} - \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

24.7 $$J_0'(x) = -J_1(x)$$

BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER n

24.8 $$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0,1,2,\ldots \\[2ex] \lim\limits_{p\to n} \dfrac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0,1,2,\ldots \end{cases}$$

This is also called *Weber's function* or *Neumann's function* [also denoted by $N_n(x)$].

For $n = 0, 1, 2, \ldots,$ L'Hospital's rule yields

24.9
$$Y_n(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\} J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(x/2)^{2k-n}$$
$$- \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\{\Phi(k) + \Phi(n+k)\}\frac{(x/2)^{2k+n}}{k!\,(n+k)!}$$

where $\gamma = .5772156\ldots$ is Euler's constant [page 1] and

24.10
$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}, \qquad \Phi(0) = 0$$

For $n = 0$,

24.11
$$Y_0(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\} J_0(x) + \frac{2}{\pi}\left\{\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) - \cdots\right\}$$

24.12
$$Y_{-n}(x) = (-1)^n Y_n(x) \qquad n = 0, 1, 2, \ldots$$

For any value $n \geqq 0$, $J_n(x)$ is bounded at $x = 0$ while $Y_n(x)$ is unbounded.

GENERAL SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION

24.13 $y = AJ_n(x) + BJ_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$

24.14 $y = AJ_n(x) + BY_n(x) \qquad$ all n

24.15 $y = AJ_n(x) + BJ_n(x)\displaystyle\int \frac{dx}{x J_n^2(x)} \qquad$ all n

where A and B are arbitrary constants.

GENERATING FUNCTION FOR $J_n(x)$

24.16
$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

RECURRENCE FORMULAS FOR BESSEL FUNCTIONS

24.17
$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

24.18
$$J_n'(x) = \tfrac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$$

24.19
$$x J_n'(x) = x J_{n-1}(x) - n J_n(x)$$

24.20
$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

24.21
$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

24.22
$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

The functions $Y_n(x)$ satisfy identical relations.

BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of sines and cosines.

24.23 $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

24.24 $$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

24.25 $$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

24.26 $$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

24.27 $$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right\}$$

24.28 $$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right\}$$

For further results use the recurrence formula. Results for $Y_{1/2}(x), Y_{3/2}(x), \ldots$ are obtained from 24.8.

HANKEL FUNCTIONS OF FIRST AND SECOND KINDS OF ORDER n

24.29 $$H_n^{(1)}(x) = J_n(x) + iY_n(x)$$

24.30 $$H_n^{(2)}(x) = J_n(x) - iY_n(x)$$

BESSEL'S MODIFIED DIFFERENTIAL EQUATION

24.31 $$x^2y'' + xy' - (x^2+n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called *modified Bessel functions of order n*.

MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER n

24.32 $$\begin{aligned} I_n(x) &= i^{-n}J_n(ix) = e^{-n\pi i/2}J_n(ix) \\ &= \frac{x^n}{2^n\Gamma(n+1)}\left\{1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} + \cdots\right\} = \sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!\,\Gamma(n+k+1)} \end{aligned}$$

24.33 $$\begin{aligned} I_{-n}(x) &= i^{n}J_{-n}(ix) = e^{n\pi i/2}J_{-n}(ix) \\ &= \frac{x^{-n}}{2^{-n}\Gamma(1-n)}\left\{1 + \frac{x^2}{2(2-2n)} + \frac{x^4}{2\cdot 4(2-2n)(4-2n)} + \cdots\right\} = \sum_{k=0}^{\infty}\frac{(x/2)^{2k-n}}{k!\,\Gamma(k+1-n)} \end{aligned}$$

24.34 $$I_{-n}(x) = I_n(x) \qquad n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots$, then $I_n(x)$ and $I_{-n}(x)$ are linearly independent.

For $n = 0, 1$, we have

24.35 $$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

24.36 $$I_1(x) = \frac{x}{2} + \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} + \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

24.37 $$I_0'(x) = I_1(x)$$

MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER n

24.38 $$K_n(x) = \begin{cases} \dfrac{\pi}{2\sin n\pi}\{I_{-n}(x) - I_n(x)\} & n \neq 0, 1, 2, \ldots \\ \lim\limits_{p\to n} \dfrac{\pi}{2\sin p\pi}\{I_{-p}(x) - I_p(x)\} & n = 0, 1, 2, \ldots \end{cases}$$

For $n = 0, 1, 2, \ldots,$ L'Hospital's rule yields

24.39 $$K_n(x) = (-1)^{n+1}\{\ln(x/2) + \gamma\}I_n(x) + \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k(n-k-1)!\,(x/2)^{2k-n} + \frac{(-1)^n}{2}\sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k) + \Phi(n+k)\}$$

where $\Phi(p)$ is given by 24.10.

For $n = 0$,

24.40 $$K_0(x) = -\{\ln(x/2) + \gamma\}I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2\cdot 4^2\cdot 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots$$

24.41 $$K_{-n}(x) = K_n(x) \qquad n = 0, 1, 2, \ldots$$

GENERAL SOLUTION OF BESSEL'S MODIFIED EQUATION

24.42 $$y = AI_n(x) + BI_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

24.43 $$y = AI_n(x) + BK_n(x) \qquad \text{all } n$$

24.44 $$y = AI_n(x) + BI_n(x)\int\frac{dx}{x I_n^2(x)} \qquad \text{all } n$$

where A and B are arbitrary constants.

GENERATING FUNCTION FOR $I_n(x)$

24.45 $$e^{x(t+1/t)/2} = \sum_{n=-\infty}^{\infty} I_n(x)t^n$$

RECURRENCE FORMULAS FOR MODIFIED BESSEL FUNCTIONS

24.46 $$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x}I_n(x)$$

24.47 $$I_n'(x) = \tfrac{1}{2}\{I_{n-1}(x) + I_{n+1}(x)\}$$

24.48 $$xI_n'(x) = xI_{n-1}(x) - nI_n(x)$$

24.49 $$xI_n'(x) = xI_{n+1}(x) + nI_n(x)$$

24.50 $$\frac{d}{dx}\{x^nI_n(x)\} = x^nI_{n-1}(x)$$

24.51 $$\frac{d}{dx}\{x^{-n}I_n(x)\} = x^{-n}I_{n+1}(x)$$

24.52 $$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x}K_n(x)$$

24.53 $$K_n'(x) = -\tfrac{1}{2}\{K_{n-1}(x) + K_{n+1}(x)\}$$

24.54 $$xK_n'(x) = -xK_{n-1}(x) - nK_n(x)$$

24.55 $$xK_n'(x) = nK_n(x) - xK_{n+1}(x)$$

24.56 $$\frac{d}{dx}\{x^nK_n(x)\} = -x^nK_{n-1}(x)$$

24.57 $$\frac{d}{dx}\{x^{-n}K_n(x)\} = -x^{-n}K_{n+1}(x)$$

MODIFIED BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of hyperbolic sines and cosines.

24.58 $$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

24.59 $$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x$$

24.60 $$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cosh x - \frac{\sinh x}{x}\right)$$

24.61 $$I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\sinh x - \frac{\cosh x}{x}\right)$$

24.62 $$I_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{\left(\frac{3}{x^2} + 1\right) \sinh x - \frac{3}{x} \cosh x\right\}$$

24.63 $$I_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{\left(\frac{3}{x^2} + 1\right) \cosh x - \frac{3}{x} \sinh x\right\}$$

For further results use the recurrence formula 24.46. Results for $K_{1/2}(x), K_{3/2}(x), \ldots$ are obtained from 24.38.

Ber AND Bei FUNCTIONS

The real and imaginary parts of $J_n(xe^{3\pi i/4})$ are denoted by $\mathrm{Ber}_n(x)$ and $\mathrm{Bei}_n(x)$ where

24.64 $$\mathrm{Ber}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \cos\frac{(3n+2k)\pi}{4}$$

24.65 $$\mathrm{Bei}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \sin\frac{(3n+2k)\pi}{4}$$

If $n = 0$,

24.66 $$\mathrm{Ber}(x) = 1 - \frac{(x/2)^4}{2!^2} + \frac{(x/2)^8}{4!^2} - \cdots$$

24.67 $$\mathrm{Bei}(x) = (x/2)^2 - \frac{(x/2)^6}{3!^2} + \frac{(x/2)^{10}}{5!^2} - \cdots$$

Ker AND Kei FUNCTIONS

The real and imaginary parts of $e^{-n\pi i/2} K_n(xe^{\pi i/4})$ are denoted by $\mathrm{Ker}_n(x)$ and $\mathrm{Kei}_n(x)$ where

24.68 $$\begin{aligned}\mathrm{Ker}_n(x) = &-\{\ln(x/2) + \gamma\} \mathrm{Ber}_n(x) + \tfrac{1}{4}\pi \,\mathrm{Bei}_n(x) \\ &+ \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} \cos\frac{(3n+2k)\pi}{4} \\ &+ \frac{1}{2}\sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,(n+k)!} \{\Phi(k) + \Phi(n+k)\} \cos\frac{(3n+2k)\pi}{4}\end{aligned}$$

24.69 $$\begin{aligned}\mathrm{Kei}_n(x) = &-\{\ln(x/2) + \gamma\} \mathrm{Bei}_n(x) - \tfrac{1}{4}\pi \,\mathrm{Ber}_n(x) \\ &- \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} \sin\frac{(3n+2k)\pi}{4} \\ &+ \frac{1}{2}\sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,(n+k)!} \{\Phi(k) + \Phi(n+k)\} \sin\frac{(3n+2k)\pi}{4}\end{aligned}$$

and Φ is given by 24.10, page 137.

If $n = 0$,

24.70 $$\mathrm{Ker}(x) = -\{\ln(x/2) + \gamma\} \mathrm{Ber}(x) + \frac{\pi}{4}\mathrm{Bei}(x) + 1 - \frac{(x/2)^4}{2!^2}(1 + \tfrac{1}{2}) + \frac{(x/2)^8}{4!^2}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) - \cdots$$

24.71 $$\mathrm{Kei}(x) = -\{\ln(x/2) + \gamma\} \mathrm{Bei}(x) - \frac{\pi}{4}\mathrm{Ber}(x) + (x/2)^2 - \frac{(x/2)^6}{3!^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots$$

DIFFERENTIAL EQUATION FOR Ber, Bei, Ker, Kei FUNCTIONS

24.72 $$x^2y'' + xy' - (ix^2 + n^2)y = 0$$

The general solution of this equation is

24.73 $$y = A\{\mathrm{Ber}_n(x) + i\,\mathrm{Bei}_n(x)\} + B\{\mathrm{Ker}_n(x) + i\,\mathrm{Kei}_n(x)\}$$

GRAPHS OF BESSEL FUNCTIONS

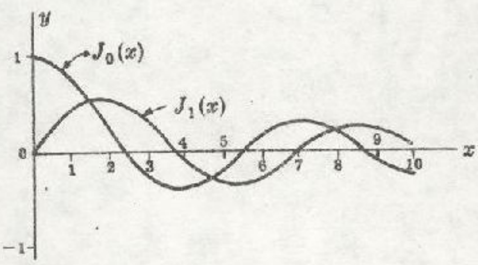


Fig. 24-1

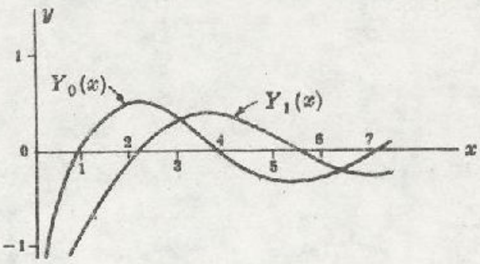


Fig. 24-2

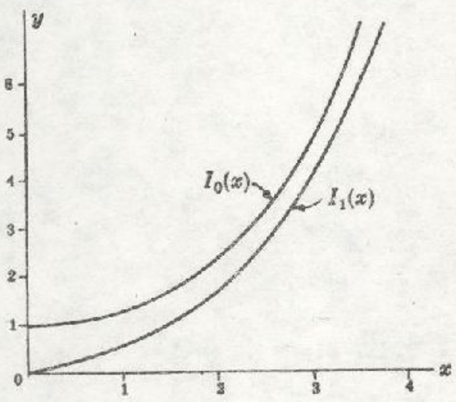


Fig. 24-3

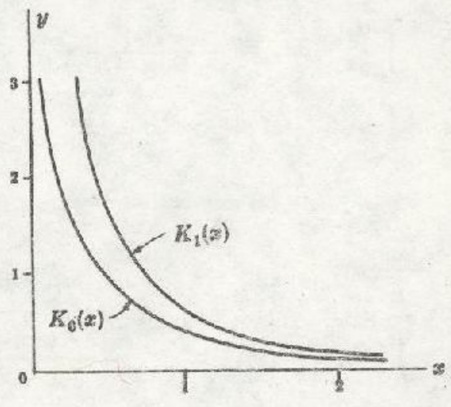


Fig. 24-4

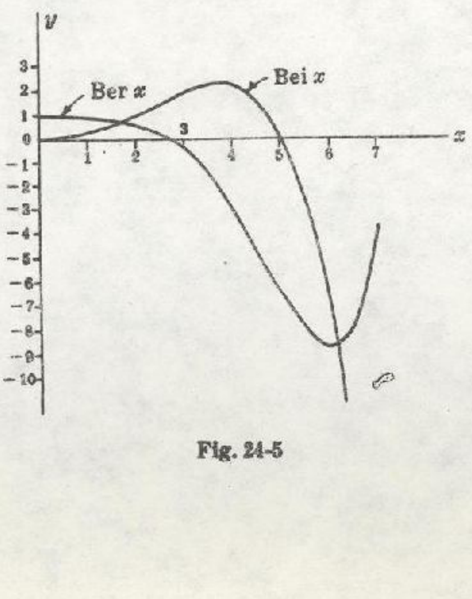


Fig. 24-5

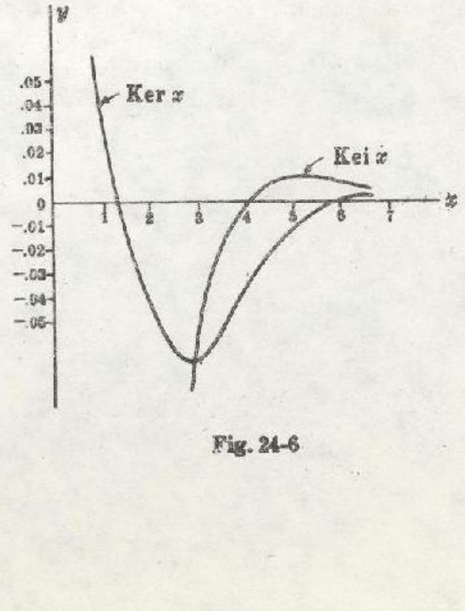


Fig. 24-6

INDEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

24.74 $\int x\, J_0(x)\, dx = x\, J_1(x)$

24.75 $\int x^2 J_0(x)\, dx = x^2 J_1(x) + x\, J_0(x) - \int J_0(x)\, dx$

24.76 $\int x^m J_0(x)\, dx = x^m J_1(x) + (m-1)x^{m-1} J_0(x) - (m-1)^2 \int x^{m-2} J_0(x)\, dx$

24.77 $\int \frac{J_0(x)}{x^2}\, dx = J_1(x) - \frac{J_0(x)}{x} - \int J_0(x)\, dx$

24.78 $\int \frac{J_0(x)}{x^m}\, dx = \frac{J_1(x)}{(m-1)^2 x^{m-2}} - \frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2} \int \frac{J_0(x)}{x^{m-2}}\, dx$

24.79 $\int J_1(x)\, dx = -J_0(x)$

24.80 $\int x\, J_1(x)\, dx = -x\, J_0(x) + \int J_0(x)\, dx$

24.81 $\int x^m J_1(x)\, dx = -x^m J_0(x) + m \int x^{m-1} J_0(x)\, dx$

24.82 $\int \frac{J_1(x)}{x}\, dx = -J_1(x) + \int J_0(x)\, dx$

24.83 $\int \frac{J_1(x)}{x^m}\, dx = -\frac{J_1(x)}{m x^{m-1}} + \frac{1}{m} \int \frac{J_0(x)}{x^{m-1}}\, dx$

24.84 $\int x^n J_{n-1}(x)\, dx = x^n J_n(x)$

24.85 $\int x^{-n} J_{n+1}(x)\, dx = -x^{-n} J_n(x)$

24.86 $\int x^m J_n(x)\, dx = -x^m J_{n-1}(x) + (m+n-1) \int x^{m-1} J_{n-1}(x)\, dx$

24.87 $\int x\, J_n(\alpha x) J_n(\beta x)\, dx = \frac{x\{\alpha\, J_n(\beta x)\, J_n'(\alpha x) - \beta\, J_n(\alpha x)\, J_n'(\beta x)\}}{\beta^2 - \alpha^2}$

24.88 $\int x\, J_n^2(\alpha x)\, dx = \frac{x^2}{2}\{J_n'(\alpha x)\}^2 + \frac{x^2}{2}\left(1 - \frac{n^2}{\alpha^2 x^2}\right)\{J_n(\alpha x)\}^2$

The above results also hold if we replace $J_n(x)$ by $Y_n(x)$ or, more generally, $A\, J_n(x) + B\, Y_n(x)$ where A and B are constants.

DEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

24.89 $\int_0^\infty e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{a^2+b^2}}$

24.90 $\int_0^\infty e^{-ax} J_n(bx)\, dx = \frac{(\sqrt{a^2+b^2} - a)^n}{b^n \sqrt{a^2+b^2}} \qquad n > -1$

24.91 $\int_0^\infty \cos ax\, J_0(bx)\, dx = \begin{cases} \frac{1}{\sqrt{a^2-b^2}} & a > b \\ 0 & a < b \end{cases}$